Chapter 2

The Platonic Solids

2.1 Polygons

Before jumping into the three-dimensional polyhedral world, we need to take a few moments to discuss polygons in two dimensions. A polygon may be determined by a list of its **vertices** (singular **vertex**). An **edge** of a polygon connects a vertex to an adjacent vertex in the list, with the convention that the last vertex in the list is connected to the first.

Now imagine a pentagon. Did you imagine only the vertices and edges of the pentagon? Or did you *also* imagine that the pentagon was filled in, including the vertices and edges of the pentagon and its interior as well? Depending on the context, either definition might prove useful. For our purposes, we will consider a pentagon (or any polygon) to include all interior points as well the vertices and edges.

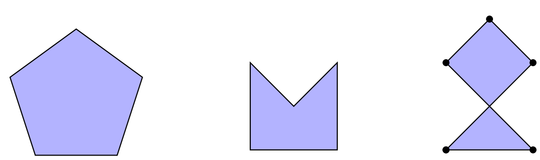


Figure 2.1: Pentagons: convex (left), nonconvex (middle), and crossed (right).

In Figure 2.1, we see three different types of pentagons. The first two types – convex and nonconvex – indicate whether or not the pentagons have any indentations. More precisely, we say that a polygon is **convex** if given any two points in the polygon, the segment joining them lies entirely in the polygon; otherwise we say that it is **nonconvex**, as illustrated in Figure 2.2.

We also say that a polygon is **crossed** or **self-intersecting** if at least one of its edges intersects another. It is important to note that the points where the edges cross are not considered vertices of the polygon. When in doubt, we will indicate the vertices by small dots (as in the rightmost pentagon in Figure 2.1). Certainly any crossed polygon is nonconvex, so the term "crossed" includes the idea of nonconvexity.

It is interesting to note that two polygons may look similar even although they have different numbers of sides. For example, in Figure 2.3, the left polygon is a crossed pentagon, while the right polygon is a nonconvex decagon.

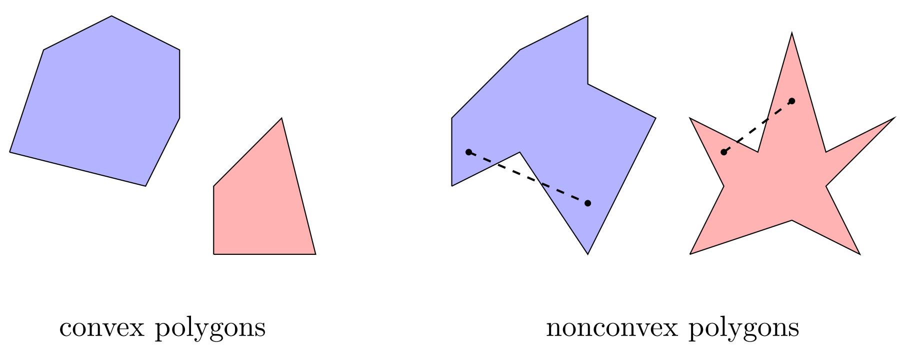


Figure 2.2: Convex and nonconvex polygons.

We will be particularly interested in a special type of polygon – a **regular** polygon is one which has all sides congruent, as well as having all angles between the sides congruent. A regular polygon may be convex, such as the left pentagon in Figure 2.1, or crossed, as shown in the left polygon in Figure 2.3 – but a regular polygon cannot be nonconvex without also being crossed.

A final technical remark is in order here. Consider the object in Figure 2.4, with the vertex list 1 – 2 – 3 – 4 – 5 – 3 – 6.

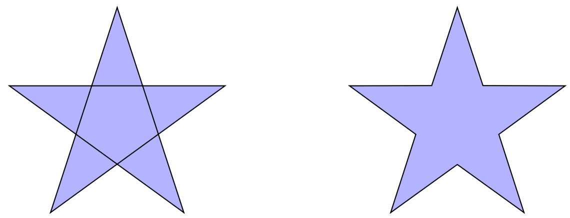


Figure 2.3: Crossed pentagon (left) and nonconvex decagon (right).

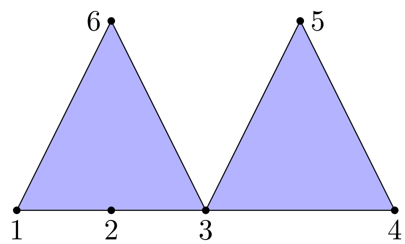


Figure 2.4: Polygon, or not a polygon?

Is this object a polygon? Should we allow three consecutive vertices to lie on a line, as with 1 – 2 – 3? Should we allow vertices to be repeated, as 3 is? While these are important issues, we will not go more deeply into them here. We will be concerned almost exclusively with two types of polygons – convex and regular. Convex polygons and regular crossed polygons (such as the pentagon in Figure 2.3) are quite well-behaved, so we will not need to further consider the subtleties of the definition of a polygon.

2.2 Polyhedra

Now that we have set our terminology in two dimensions, we look at extending these ideas into three dimensions. It turns out that this task is not nearly as straightforward as it is in the plane. Peter Cromwell, in his book *Polyhedra,*[1] gives a lengthy discussion of the history of the attempt to rigorously define what a polyhedron is. There is still no universally accepted definition.

To give you a sense of why it is difficult to come to consensus, consider the object in Figure 2.5. It is easy to construct – just drill a square hole through a prism. Would you call it a polyhedron? What about the top and bottom faces? Are they polygons? We would need to modify the definition given above, since using just one vertex list would not be sufficient to describe these two faces. At different times in the historical development of the concept of a polyhedron, sometimes this object was considered a polyhedron, and at other times it wasn't.

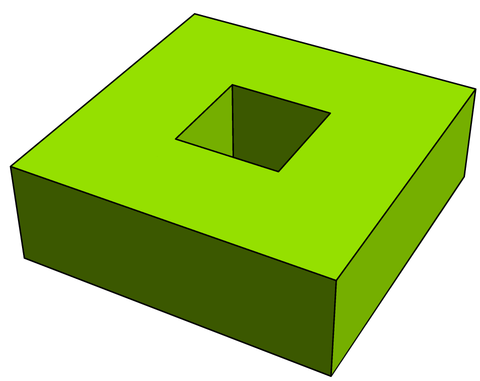


Figure 2.5: Polyhedron, or not a polyhedron?

There are further issues to think about as well. Consider the object in Figure 2.6. One way to look at the object is that it is composed of 60 isosceles triangular faces. But it is *also* possible to think of this object as having 12 pentagonal faces (one of which is shown in orange on the right), each intersecting five others. Which is it? How the object is considered determines what properties it has. Poinsot (1777–1859) was the first known

[1]Cromwell, Peter. *Polyhedra.* New York: Cambridge University Press. 1997.

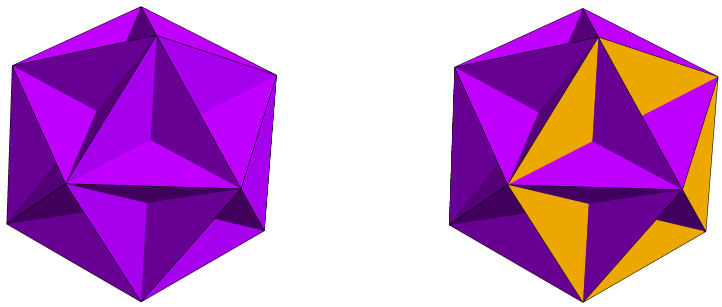


Figure 2.6: Great dodecahedron.

to describe this object – the **great dodecahedron** – which he considered to have 12 pentagonal faces.[2]

The definition of convexity, however, does extend easily to three dimensions. An object in three dimensions is said to be **convex** if given any two points in the object, the segment joining them lies entirely within the object. As a result, it *is* possible to give a precise definition of a convex polyhedron – but the nonconvex case is still problematic.

The simplest way to define a convex polyhedron is by specifying its vertices. So if $\mathcal{V}$ is a finite set of points, we define the **convex hull** of $\mathcal{V}$ to be the smallest convex set which includes members of $\mathcal{V}$. Two illustrations of convex hulls in two dimensions are shown in Figure 2.7.

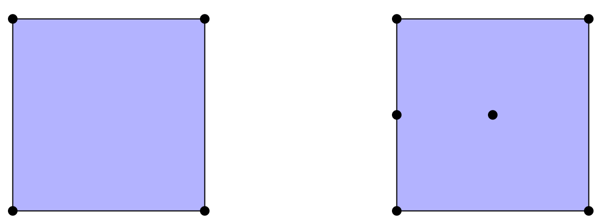


Figure 2.7: Convex hulls of two point sets.

Note that the convex hulls are both the same here – because a square does in fact have four vertices, throwing in the extra two points in the right square shown in Figure 2.7 does not change the convex hull. These additional points

[2]Cromwell, pp. 251–2.

are not vertices of the square, and leaving one or both of them out of the set will not change the convex hull.

But in the square on the left, if you took out any one of the points in the set, the convex hull would change – it would no longer be a square, but a right triangle. When this happens – when taking out any one of the points of $\mathcal{V}$ changes the convex hull – we say that $\mathcal{V}$ is **minimal.**

Now we are in a position to define a convex polyhedron. So suppose a finite set $\mathcal{V}$ of points in three dimensions is given, and those points do not all lie in a plane. If $\mathcal{V}$ is minimal, then we call the convex hull of $\mathcal{V}$ a **convex polyhedron** with **vertex set** $\mathcal{V}$. We call the segments joining two adjacent vertices of a polyhedron **edges** of the polyhedron, and polygons on the boundary of a polyhedron **faces** of the polyhedron. You should be able to easily count that a cube has 8 vertices, 12 edges, and 6 faces.

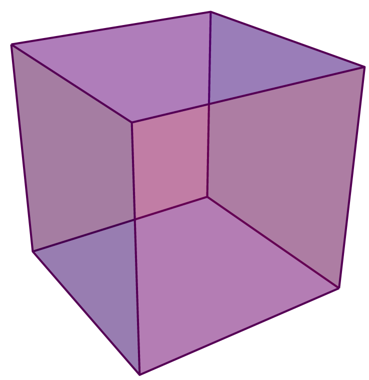


Figure 2.8: Cube.

It is possible to more rigorously define edges and faces of a polyhedron, but we will not do so here. It is important to see the subtleties in creating a definition for a polyhedron – but the definition of a convex polyhedron here, together will a little geometrical intuition, will be entirely adequate for our purposes.

Interestingly, Poinsot considered the great dodecahedron to be a convex polyhedron since the angles between the pentagonal faces are less that 180°,[3] although we would not do so today.

[3] Cromwell, p. 252

As a final remark, it is possible to use the concept of a convex hull to define polygons as well (see Figure 2.7). However, despite the issues described at the end of Section 2.1, the more usual practice to is define a polygon by a list of its vertices.

2.3 The Platonic Solids

The most symmetrical convex polygons in the plane are those that are regular. What are the most symmetrical convex polyhedra in three dimensions?

We think of a cube as a highly symmetrical polyhedron. What gives it such symmetry? Well, you immediately notice that all the faces are squares, which are regular polygons. Another observation is that exactly three squares meet at each vertex of the cube.

Are there other polyhedra with similar properties? Yes, and we'll enumerate them all. But first, a definition. A **Platonic solid** is a polyhedron with the following properties:

(P_1) *It is convex.*

(P_2) *Its faces are congruent regular polygons.*

(P_3) *The same number of polygons meet at each vertex of the polyhedron.*

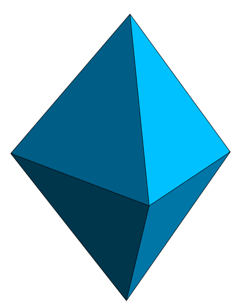


Figure 2.9: Triangular bipyramid.

Note that since a Platonic solid is convex, the polygons referred to in (P_2) must also be convex. Also note that it is possible to satisfy (P_2) without satisfying (P_3), as illustrated with the triangular bipyramid in Figure 2.9. Three triangles meet at two of the vertices, while four triangles meet at the other three.

Given this definition, a natural question arises: just how many Platonic solids are there, and what do they look like? We'll answer this question in two different ways in a moment: geometrically and algebraically.

As a final remark, the Platonic solids are so named since the earliest written reference to these solids was Plato's *Timaeus*. We are certain that Plato was not the first one to discover these polyhedra, since some Etruscan charms with a dodecahedron-like shape were found which dated back to 500 B. C.[4]

[4]Cromwell, p. 71.

2.4 Thinking Geometrically

Since the faces of any Platonic solid are convex regular polygons, we begin by considering which solids have triangles for faces, then squares, and so forth. By carefully examining all cases, we'll soon arrive at a complete enumeration.

Let's consider Platonic solids with triangular faces first. The fewest number of triangles which can meet at a vertex is three. It's easy to construct a polyhedron with three triangles meeting at each vertex – just take a triangular pyramid. Four triangles are required (see Figure 2.10(a)), and hence this polyhedron is called a **tetrahedron;** the Greek prefix for four is "tetra- ." In general, the name of a Platonic solid is derived from the Greek prefix for the number of faces on the Polyhedron, along with the suffix "-hedron," for "face."

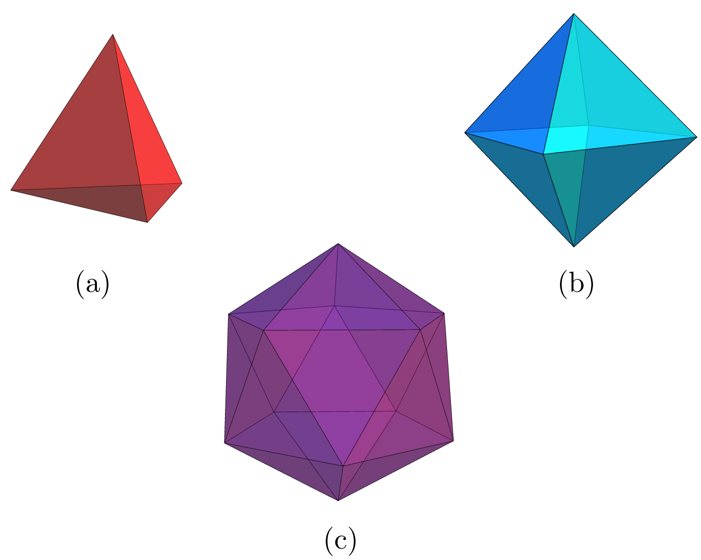


Figure 2.10: Platonic solids with triangular faces.

What about four triangles meeting at a vertex? We know that four triangles meet at the apex of a square pyramid, while two triangles meet at each vertex of the square base. This means that if we take two square pyramids and join them base-to-base, the squares disappear, leaving $2+2=4$ triangles at each vertex of the interior squares. The result is a convex polyhedron with four equilateral triangles meeting at each vertex. Since two pyramids were used, $2 \times 4 = 8$ triangles are needed (see Figure 2.10(b)), and so this polyhedron is called an **octahedron.**

Five triangles at a vertex is a bit trickier. All in all, twenty triangles are required (see Figure 2.10(c)), and so we call this polyhedron an **icosahedron** ("icosi" is the Greek prefix for "20"). Let's look at the icosahedron in more detail.

Suppose we try at first to extend our base-to-base square pyramid construction of an octahedron to a base-to-base pentagonal pyramid construction of a polyhedron with five triangles meeting at each vertex. Of course five triangles meet at the apex (and the vertex opposite) but, as with the square pyramid, only four triangles meet at each vertex of the disappearing pentagonal bases (see Figure 2.11(a)). Because of its construction, the polyhedron in Figure 2.11(a) is called a **pentagonal bipyramid**.

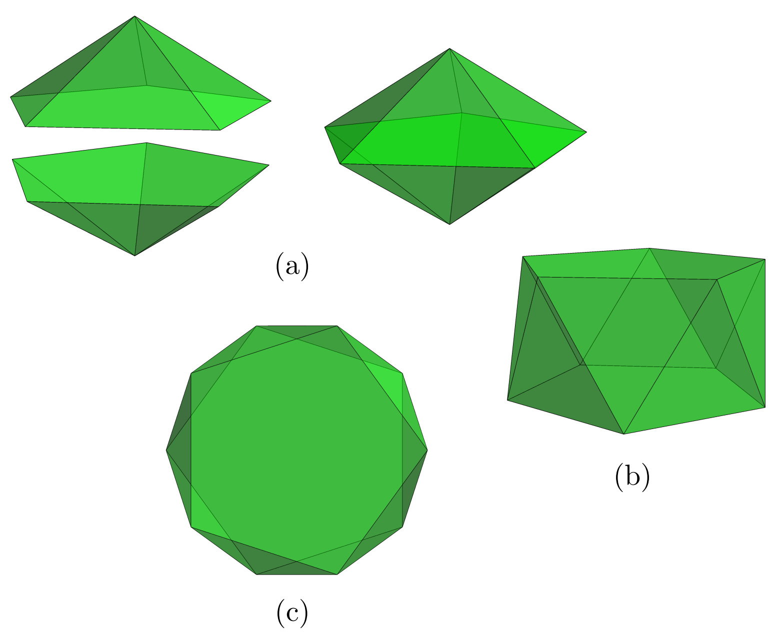
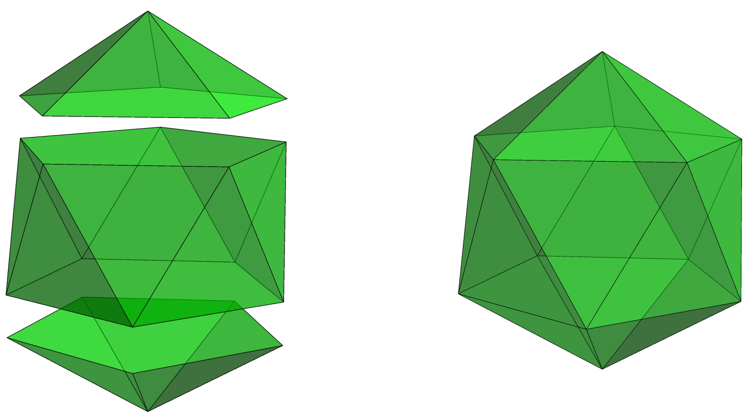


Figure 2.11: Toward building an icosahedron.

But recall that Platonic solids must have the *same* number of faces meeting at each vertex. It turns out there's a fairly simple way to remedy this situation. Consider the pentagonal toy drum of Figure 2.11(b). The top and bottom pentagons are out of phase by 36° (see Figure 2.11(c) for a top view of Figure 2.11(b)), and the space between is filled by a zig-zag of ten equilateral triangles. Because the pentagons are out of phase, this polyhedron is called a **pentagonal antiprism.** The important feature here is that exactly three triangles (and one pentagon) meet at each vertex. As a result, putting a pentagonal pyramid on both the top and bottom of this antiprism results in a polyhedron with precisely five triangles meeting at each of its twelve vertices (see Figure 2.12). The pentagonal antiprism contributes ten equilateral triangles, and each of the pentagonal pyramids contributes five, for a total of twenty triangles – hence the name "icosahedron."



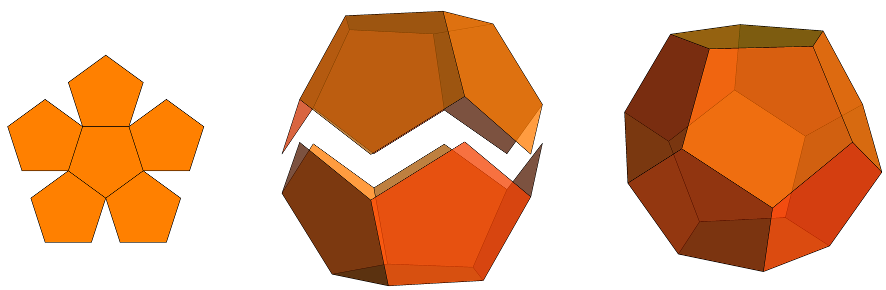
(a) (b)

Figure 2.12: The icosahedron.

Our search for Platonic solids with triangular faces ends here, for we see that if six equilateral triangles were to meet at a vertex, the angles would sum to 360°, and hence any vertex with six equilateral triangles would be flat. Of course this doesn't result in a convex polyhedron, but rather a tiling of the plane by equilateral triangles.

It is clear we need to stop here; adding more equilateral triangles at a vertex only makes the situation worse. This observation isn't confined simply to triangles, though – in fact, for *any* convex polyhedron (Platonic solid or not), the angles of the polygons meeting at any vertex must sum to *less* than 360°. So when considering other regular polygons, it will be easy to know when we can stop and move on to the next polygon.

Now let's consider Platonic solids with square faces. This case is easy – we already have the cube, and since $4 \cdot 90° = 360°$, there are no more Platonic solids with square faces. Four squares would be flat – you've certainly seen floors covered with square tiles meeting four at a vertex.



(a) (b) (c)

Figure 2.13: Building a dodecahedron.

What about regular pentagons? It turns out that there is a Platonic solid with three pentagons at each vertex. Perhaps the best way to imagine such a solid is to begin with an arrangement of six pentagons as shown in Figure 2.13(a). Five of these pentagons may be folded up to yield a bowl-like shape. As it happens, two such bowls fit exactly together. The result, as it requires twelve pentagons, is called a **dodecahedron** (or sometimes a **pentagonal dodecahedron**).

While the descriptions of the other Platonic solids seem fairly straightforward, you might not be so convinced that the two bowls in Figure 2.13(b) fit *exactly* together. Sure, it looks like it in the picture – but looks may be deceiving. In fact, quite a few near misses are known – that is, polyhedra which look like their faces are all regular polygons, but in fact some are off by a little bit.

Convex polyhedra whose faces are *all* regular polygons are called **Johnson solids,** after the mathematician who first described all 92 of them and con-

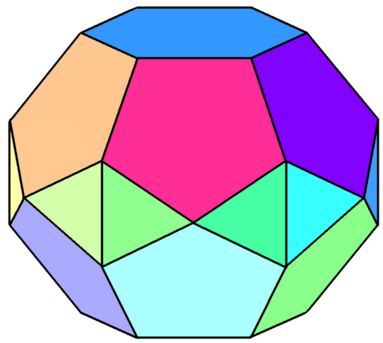


Figure 2.14: Near miss polyhedron.

jectured that the list was complete;[5] a year later, a proof of the completeness was actually found.[6]

For example, the polyhedron shown in Figure 2.14 *looks* like it's made of two hexagons (top and bottom), two rings of five pentagons, with twelve equilateral triangle filling in the gaps. But it can be shown mathematically that if all the faces are perfectly flat, then they cannot fit together *exactly.* Some will have to be slightly bent. Or alternatively, since this polyhedron is not in the list of Johnson solids, it *must* be a near miss. Several other near-miss polyhedra are known.[7]

So how do we know the bowls in Figure 2.13 fit together exactly? One way is to make a *mathematical* model of the dodecahedron. In other words, we extend the idea of a Cartesian coordinate system to three dimensions, assign x, y, and z coordinates to the vertices, and then algebraically *prove* that all faces are exactly regular pentagons. Nothing left to chance. Rather than pursuing a long digression right now, we'll address this issue later on in Chapter 9.

Let's get back to enumerating the Platonic solids. Now each angle of a regular pentagon has measure 108°. Hence four such angles have measure

[5] Johnson, Norman W. (1966). "Convex Solids with Regular Faces." Canadian Journal of Mathematics. 18: 169–200.

[6] Zalgaller, Victor A. (1967). "Convex Polyhedra with Regular Faces." Zap. Nauchn. Semin. Leningr. Otd. Mat. Inst. Steklova (in Russian). 2: 1–221.

[7] https://en.wikipedia.org/wiki/Near-miss_Johnson_solid

$432° > 360°$, and so it is impossible to create a vertex of a convex polyhedron with four (or more) pentagons at a vertex.

Next, we consider Platonic solids with regular hexagonal faces. With three hexagons at each vertex we are already flat – this is just a hexagonal tiling of the plane. As a result, there are no Platonic solids with regular hexagonal faces.

Our search stops here. Since three hexagons result in a flat vertex, three regular polygons with more than six sides, if they met at a vertex, would contribute more than 360°. As with the case of four pentagons, no convex polyhedron may be formed.

So we've found all five Platonic solids – the tetrahedron, octahedron, icosahedron, cube (which is sometimes called the **hexahedron**), and dodecahedron. These polyhedra are the base forms in three dimensions on which most other highly symmetric polyhedra are based, so we do well to become familiar with them.

2.5 An Algebraic Enumeration

We just proved – geometrically – that there are only five Platonic solids. As is often the case in geometry, by looking at the problem from an *algebraic* perspective, we can also find a solution. This is what we're about to do – but first we need an important relationship known as **Euler's formula.** Euler's formula states that if V denotes the number of vertices of a convex polyhedron, E the number of edges, and F the number of faces, then

$$V - E + F = 2. \tag{2.1}$$

A cube, for example, has eight vertices, twelve edges, and six square faces – and $V - E + F = 8 - 12 + 6 = 2$. It would be a good exercise to take a moment now and verify that Euler's formula is valid for the other Platonic solids as well.

We won't be proving Euler's formula here. In *Euler's Gem,*[8] David Richeson provides a rich history of Euler's formula, along with proofs by Euler

[8]Richeson, David S. *Euler's Gem.* Princeton: Princeton University Press. 2008.

himself, Legendre (my personal favorite), and Cauchy. He also addresses the following interesting question: are there polyhedra which satisfy Euler's formula which are *not* convex? Again, there is a rich history of this inquiry; Richeson devotes an entire chapter to those who have considered this question.[9]

How can we use Euler's formula? We need to go back to the definition of a Platonic solid given in (P_1)–(P_3). Although each of these is a *geometric* statement, we've got to find a way to say each statement *algebraically.* To help with this, we introduce a few more variables: we let p stand for the number of sides on each face (since all faces are the same regular polygon), and q stand for the number of faces that meet at each vertex (since the same number meet at each vertex).

The algebraic analogue of (P_1) is just Euler's formula:

$$(P_1') \quad V - E + F = 2.$$

Here, convexity is the key idea – if a polyhedron is not convex, there is no guarantee that Euler's formula is valid (although sometimes it is). We should also point out that convexity is the *only* requirement for Euler's formula to be valid – there are no assumptions about faces or vertices. So even though a square pyramid is not a Platonic solid, it *is* convex, so we may verify Euler's formula: $5 - 8 + 5 = 2$.

What about (P_2)? This states that all faces are the same regular polygon. We can adapt this by counting edges – since there are F faces, and p edges on each face, the polygons contribute pF edges to the Platonic solid. But when we build a Platonic solid, we join polygons edge-to-edge – so at each edge of the Platonic solid, *two* edges from the polygons meet. This means that the number of edges on the polygons is in fact twice the number of edges on the Platonic solid, or

$$(P_2') \quad pF = 2E.$$

We illustrate this in Figure 2.15. You can clearly see that on the six squares ($p = 4$ and $F = 6$), there are $pF = 24$ edges. But as they come together, they meet in pairs – and therefore pF overcounts the edges by a factor of 2.

[9] Richeson, pp. 145–155.

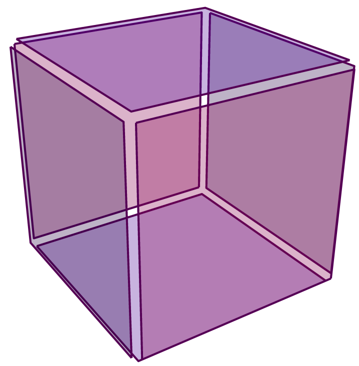


Figure 2.15: Squares meeting two to an edge.

We use the same idea with (P_3). Here, we see that q faces meet at each vertex. But if you think about it for a moment, this also means that q *edges* meet at each vertex. So V vertices with q edges meeting at each one gives a total of qV edges – almost. Notice that an edge connects two vertices, so that qV counts each edge *twice* – once for each vertex the edge connects. Translating this into an algebraic statement, we have

$$(P_3') \quad qV = 2E.$$

How should we proceed with (P_1')–(P_3')? Note that we have three equations, but five variables – but here's the catch. Each variable must be a *positive integer.* Usually, when you have more variables than equations, you have infinitely many solutions. But when you limit the values the variables might take, it might be that the number of solutions decreases – and may even be finite.

We can limit the values of the variables even further. In our case, we see that not only is p a positive integer, but $p \geq 3$. This is because the faces of the Platonic solids are polygons, which must have at least three sides. Further, $q \geq 3$ as well, since in order to create a three-dimensional polyhedron, at least three faces must meet at each vertex. All these restrictions will limit the number of solutions to (P_1')–(P_3') – in fact, we should find exactly five solutions, one corresponding to each Platonic solid.

We begin by solving for V and F from (P_3') and (P_2') and substituting into

(P_1'), which gives us

$$\frac{2E}{q} - E + \frac{2E}{p} = 2.$$

A little algebra yields

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{2} + \frac{1}{E}, \tag{2.2}$$

which must therefore be valid for any Platonic solid.

To solve (2.2), we'll start by observing that since E is a positive integer, then $E > 0$, and so $1/E > 0$ as well. This means that

$$\frac{1}{p} + \frac{1}{q} > \frac{1}{2}.$$

Now multiply through by $2pq$ and rearrange terms:

$$pq - 2q - 2p < 0.$$

The trick here is to add 4 to each side so that the left-hand side factors, which results in

$$(p-2)(q-2) < 4. \tag{2.3}$$

Solving (2.3) is now fairly simple, given the restrictions on p and q. Recall that p and q are positive integers, and both are at least 3. Then $p - 2 \geq 1$ and $q - 2 \geq 1$, which means that $p - 2$ and $q - 2$ are positive integers whose product is less than 4.

This drastically cuts down the number of solutions. In fact, we know that p or q (or possibly both) must be 3. Because if p and q were both at least 4, then $p - 2$ and $q - 2$ would both be at least 2, meaning their product would be 4 or more – which directly contradicts (2.3).

When $p = 3$, q can only be 3, 4, or 5, since if q is 6 or more, then we would have $(p-2)(q-2) \geq 4$. Symmetrically, when $q = 3$, p can only be 3, 4, or 5. Note that the case $p = 3$ and $q = 3$ is counted twice, so there are just five solutions, as we predicted.

These five solutions are summarized in Table 2.1.

p	q	V	E	F	Platonic Solid	$\{p, q\}$
3	3	4	6	4	Tetrahedron	$\{3, 3\}$
3	4	6	12	8	Octahedron	$\{3, 4\}$
4	3	8	12	6	Cube	$\{4, 3\}$
3	5	12	30	20	Icosahedron	$\{3, 5\}$
5	3	20	30	12	Dodecahedron	$\{5, 3\}$

Table 2.1

Here, we include one row for each possibility. Once we have p and q, we can find E from (2.2). Then knowing E, we may calculate F and V from (P_2') and (P_3'). Thus, we get the same enumeration as we did using a purely geometrical argument.

The polyhedron with regular faces of p sides, where q meet at each vertex, is sometimes denoted by $\{p, q\}$. This notation for referring to a polyhedron is called a **Schläfli symbol,** and is a very convenient notation – we will have occasion to use it in later chapters. In addition to his other achievements, Schläfli was one of the earliest to show how Euler's formula may be generalized to dimensions higher than three.[10]

[10] Cromwell, p. 247.