Making Calculus Accessible

2023 Joint Annual Meetings

MAA-FL/FTYCMA

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Making Calculus Accessible

Mainstream calculus texts are unreadable by many students in the life sciences. So why not write a textbook which:

- Emphasizes concepts over algebraic manipulation,
- Uses informal language rather than precise mathematical language,
- Illustrates extensively with graphs (static and interactive),
- Organizes concepts pedagogically rather than formally.

In addition, all homework problems include complete solutions.

Concepts over Algebra

From Stewart:

Find the critical numbers of $g(x) = x^{4/5}(x-4)^2$.

Find $\dfrac{d}{dx}\cos(\sqrt{\sin(\tan(\pi x))})$.

Find $\dfrac{d}{dx}(x + (x + \sin^2(x))^3)^4$.

Caveats

- Based on last semester, so a few topics are missing due to classes cancelled because of hurricane Ian (all students were evacuated).
- Almost all students are life science majors, so the text would need to be supplemented for physical science majors.
- Calculus is needed for grad school applications, not the major courses.

This talk is available on my website; details at the end.

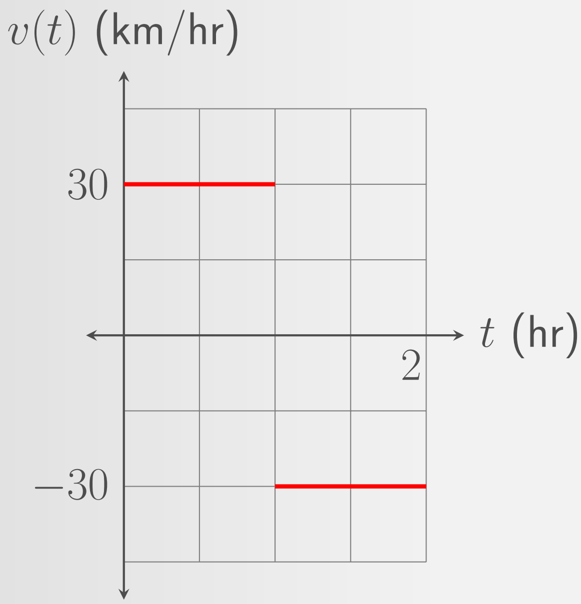
Day 1: Algebra Review

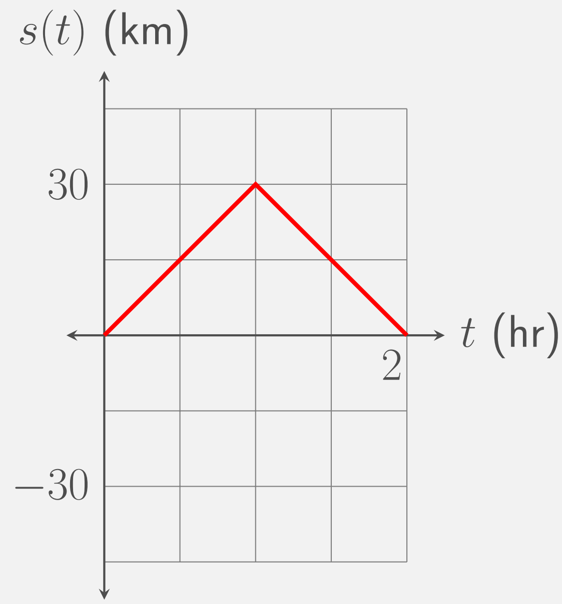
Topics to include:

- Simple factoring,
- Rationalizing numerators and denominators,
- Fractional exponents,
- Finding common denominators,
- Simplifying complex fractions,
- Converting from degrees to radians, and vice versa,
- Reviewing the unit circle.

Good to float around the room and see what students' skills are like.

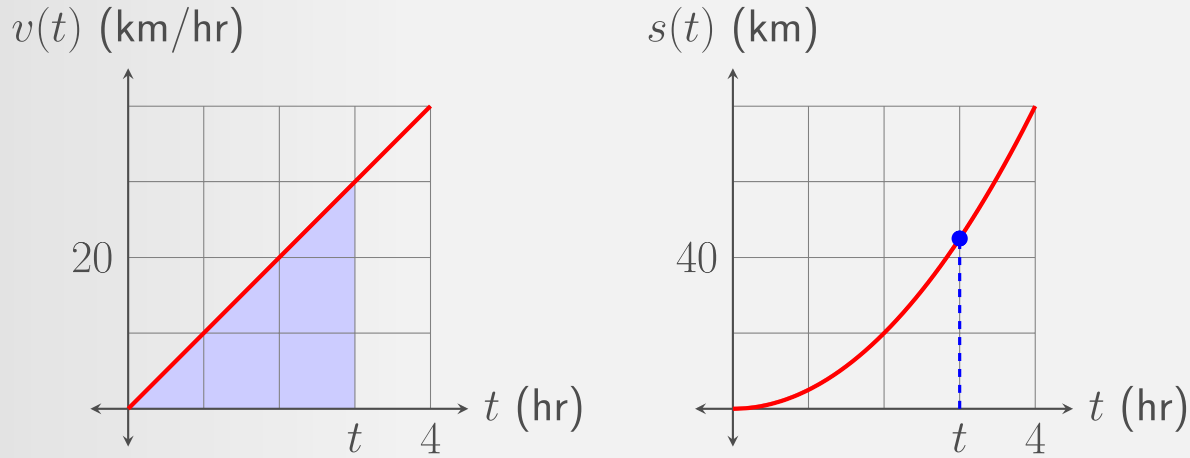
Day 2: Introduction to Physics





- Most students don't understand velocity and displacement.
- Displacement can be found using a velocity graph, but *not* by using a speed graph.

Day 3: What Calculus is All About



Velocity graph (left), and displacement graph (right).

- Motivate right away, rather than later.
- Introduce negative areas.
- Illustrate the Fundamental Theorem of Calculus with velocity and displacement.

Day 4: Definition of the Derivative

- No need to wait!
- Use desmos for secant line demonstration.
- Motivate limits by saying:
  - you need a slope since you only have one point on the tangent line,
  - you can't just substitute $h = 0$, or you'll get $\frac{0}{0}$.
- When the h cancels – as with polynomials, simple rational functions, and square roots – the limits are simple substitution.

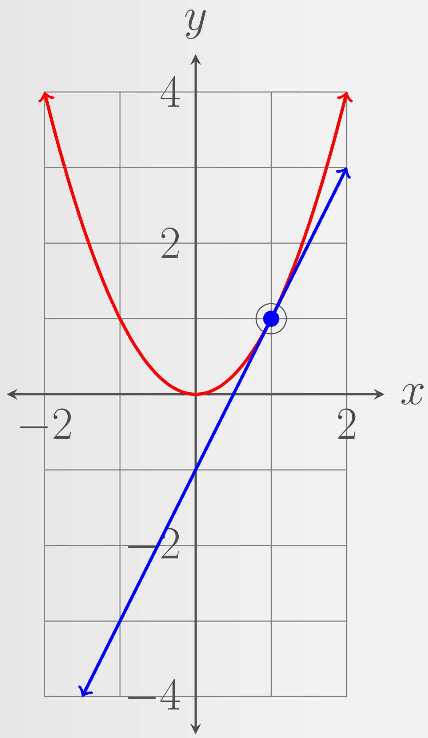
Day 5: Derivative of $\sin(x)$

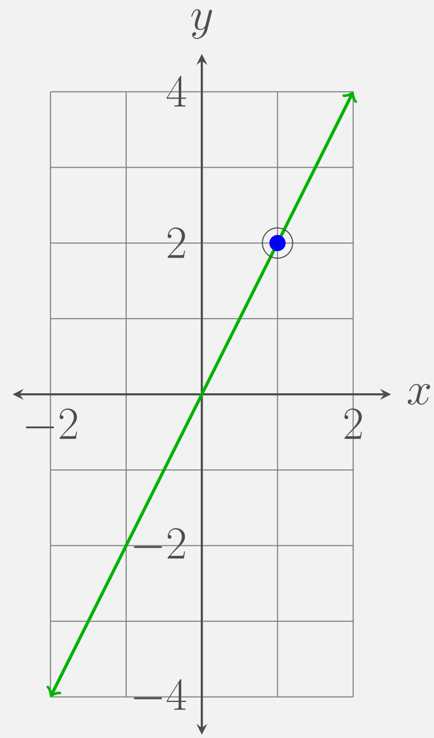
We need $\displaystyle\lim_{h\to 0}\frac{\sin(h)}{h}$ and $\displaystyle\lim_{h\to 0}\frac{\cos(h)-1}{h}$.

- Important since the h does not cancel,
- Show how to evaluate these limits numerically and graphically, *not* geometrically,
- Therefore the Squeeze Theorem is not needed.

Limits can be approached purely algebraically, numerically, or graphically. If numerical and graphical limits are the same, we're good to go.

Day 6: Geometry of the Derivative





- Increasing/decreasing, local maxima/minima.
- Graphs, graphs, and more graphs!

Day 7: Rules of Differentiation

- Usual algebraic rules,
- Power Rule,
- Product Rule,
- Quotient Rule,
- Chain Rule.

My students took math during the pandemic, and skills are often not strong. Lots of examples here, no complex algebra.

The day before, I gave homework on writing a function as the composition of two functions.

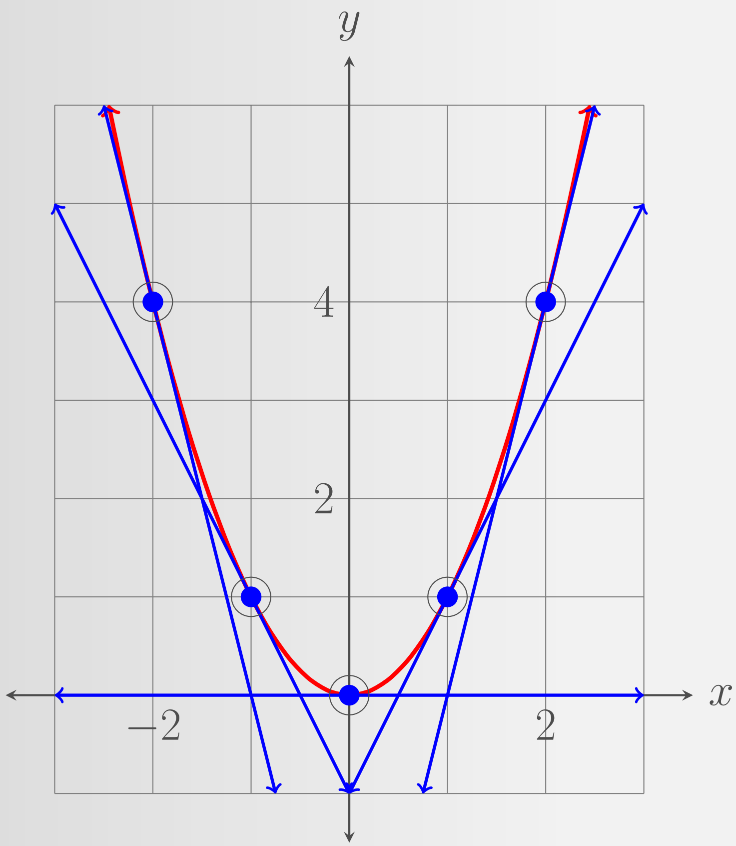
Day 8: Using Differentiation Rules

Examples:

- $\dfrac{x^4 - 3x^2 + 5x}{x}$
- $\dfrac{\cos(x)}{x^{-2}}$
- $(x^2 + 1)(x - 3)$
- $\dfrac{5}{x^6}$
- $x^3\sqrt{x}$

Rewriting first is very challenging. Use the second half of the class for in-class work to see how students are doing.

Day 9: Geometry of the Second Derivative

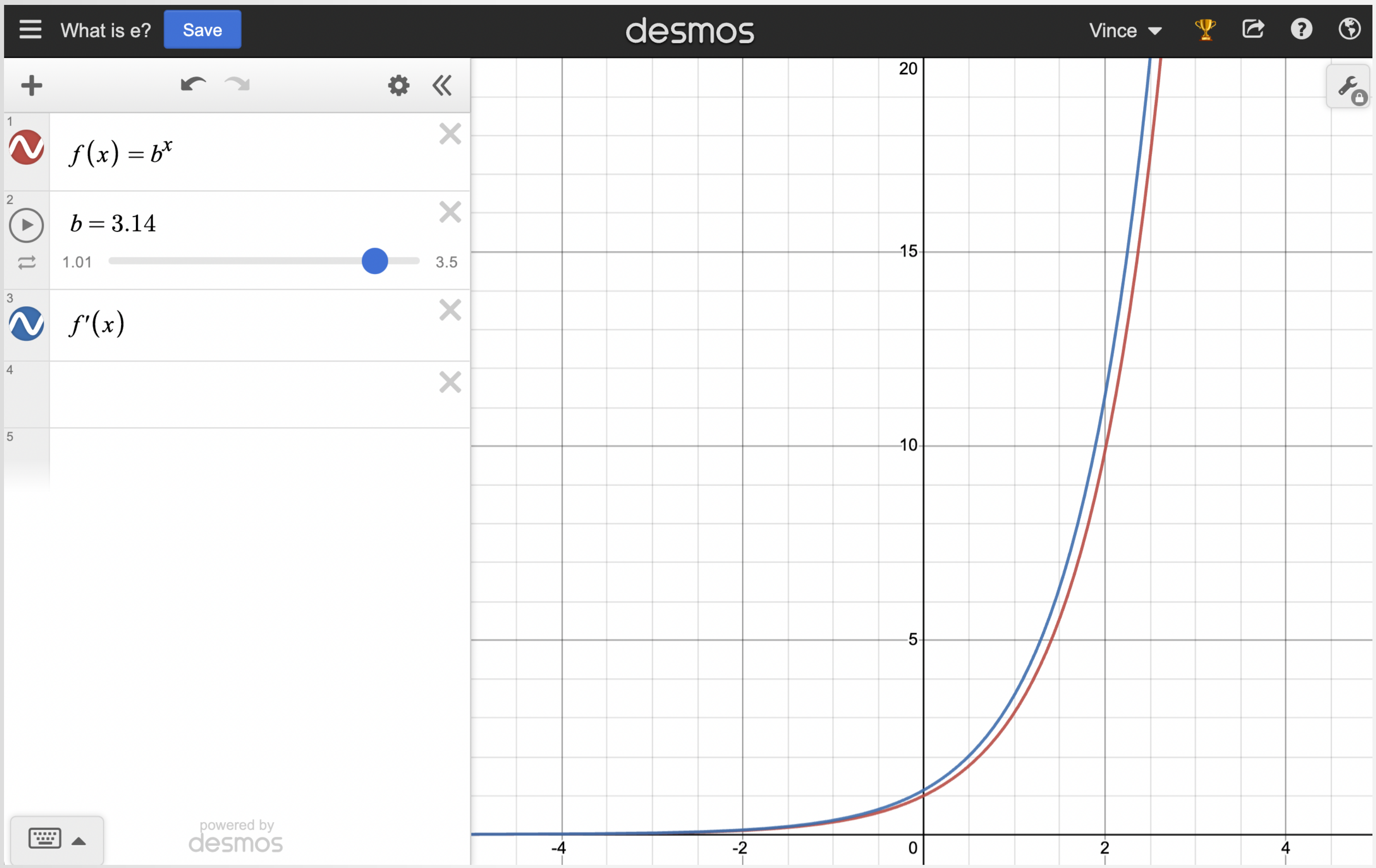


x	$f'(x)$	Eqn. of tangent
-2	-4	$y = -4x - 4$
-1	-2	$y = -2x - 1$
0	0	$y = 0$
1	2	$y = 2x - 1$
2	4	$y = 4x - 4$

Show that $f'(x)$ is increasing. Lots of graphs.

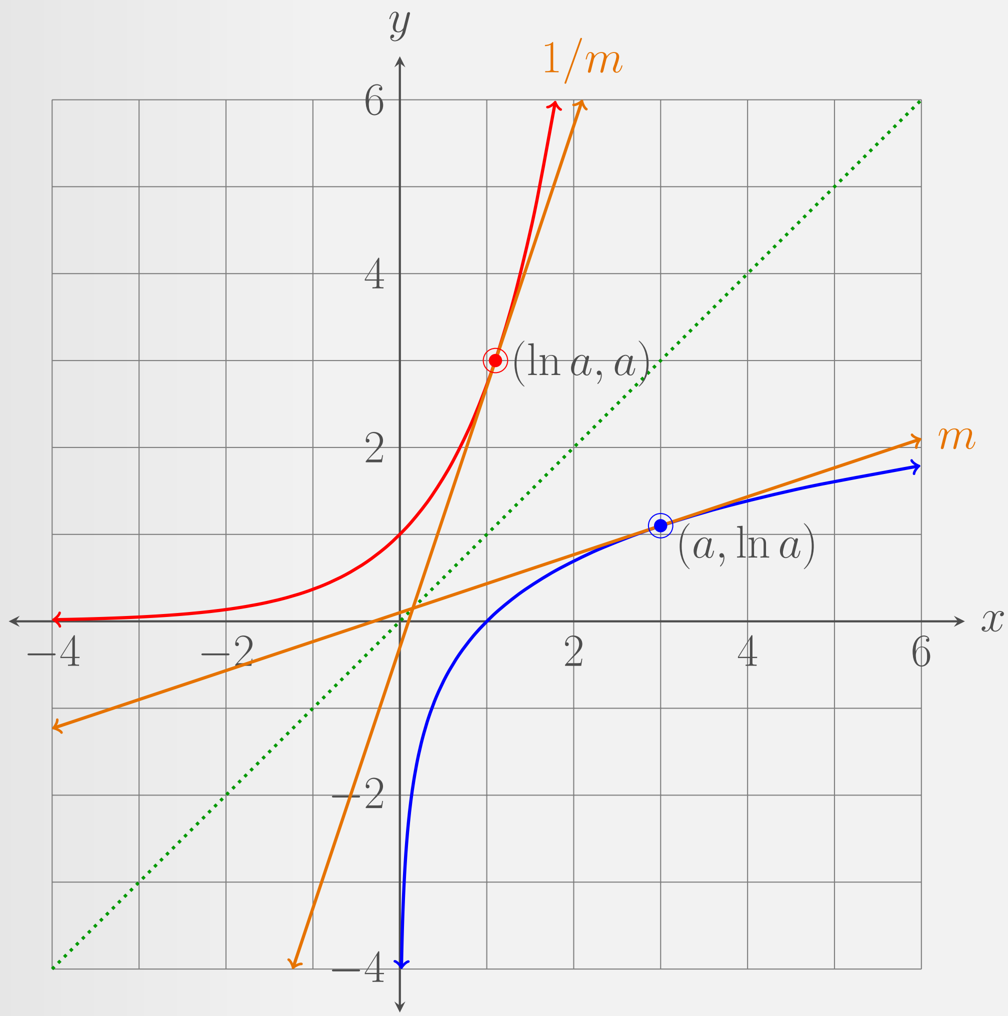
Day 10: Exponential Functions

Motivate $\frac{d}{dx}e^x = e^x$ with desmos. Use bacterial growth.

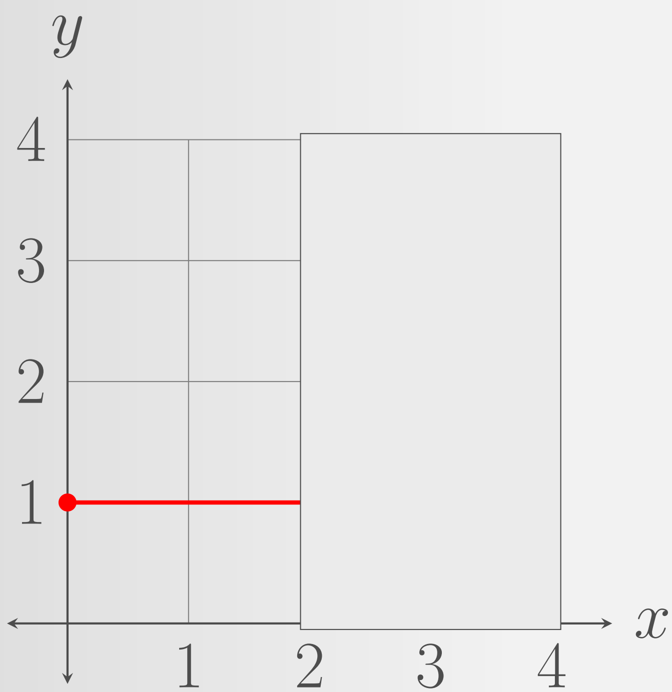


Day 11: Natural Logarithms

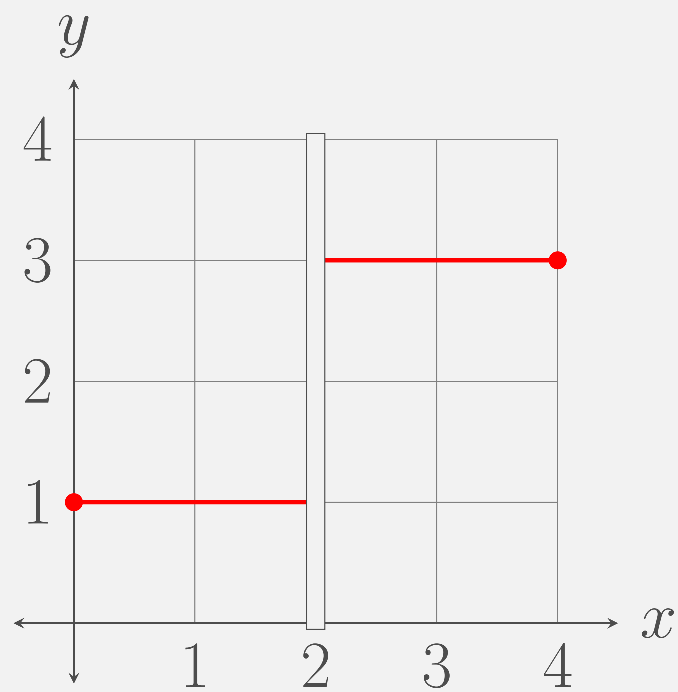
Review rules of logarithms.



Day 12: Limits and Continuity



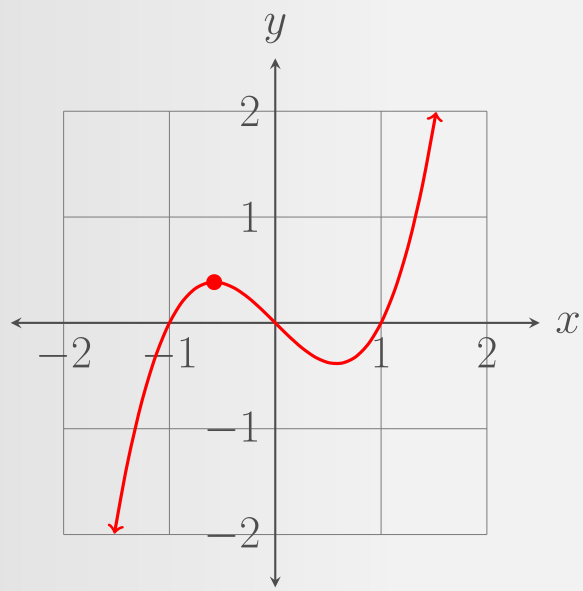
$$\lim_{x\to 2^-} f(x)$$

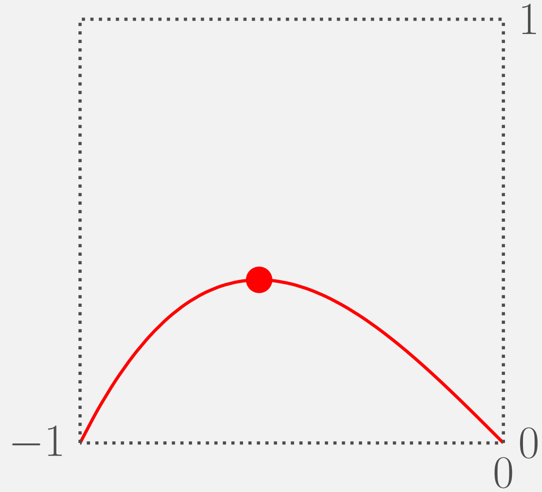


$$\lim_{x\to 2} f(x)$$

Continuity, essential/removable discontinuities. Introduce limits as needed – to describe graphs.

Day 13: Optimization





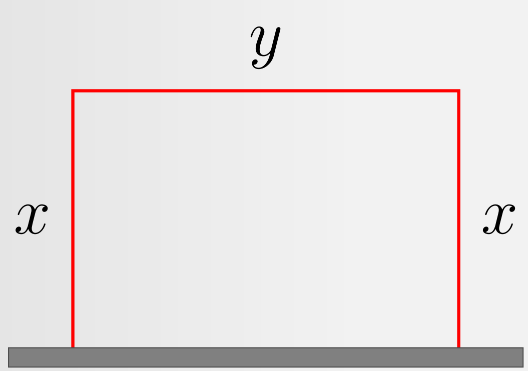
Suppose a function $f(x)$ is defined on a closed interval $[a, b]$ and is continuous. Then both a global minimum and a global maximum exist. To find them:

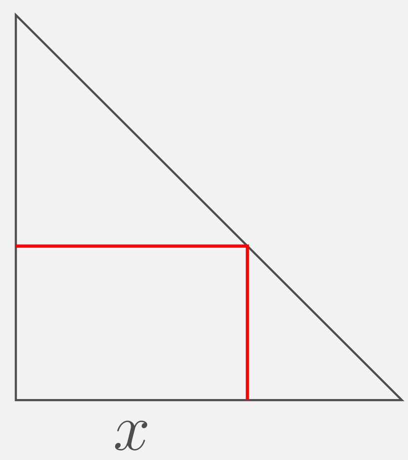
1. Determine where $f'(x) = 0$ or $f'(x)$ does not exist,
2. Evaluate $f(x)$ at these points and the endpoints a and b,
3. Select the lowest and highest values among these values.

Day 14: Extreme Value Theorem

- Suppose two positive numbers sum to 10. What is the largest their product can be?

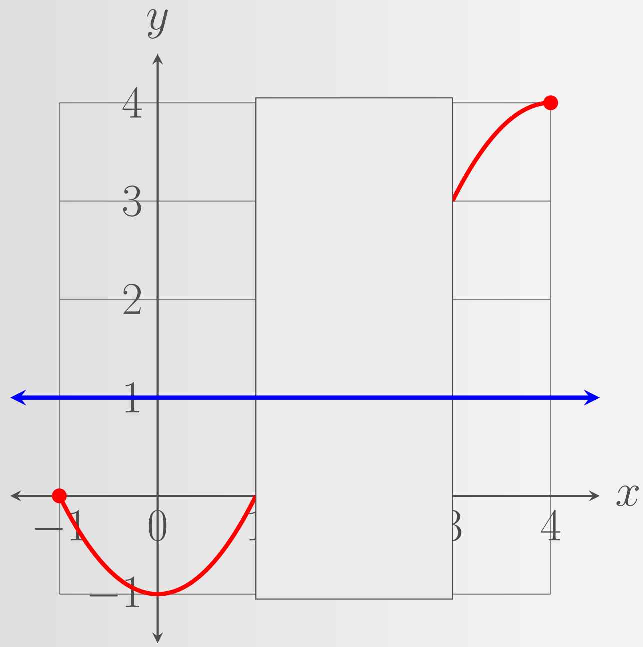
- Suppose you are given a positive number. First, take the square root. Then add 3. Finally, subtract the given number. What is the largest number you can get?

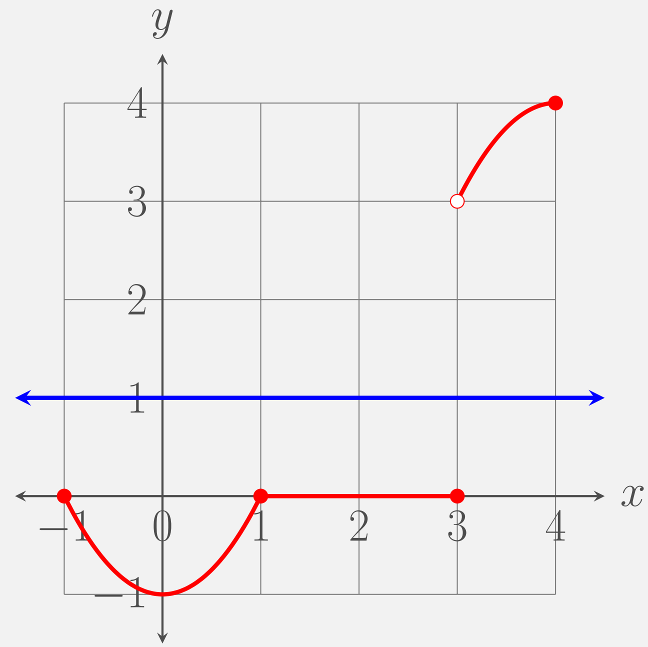




Limit types of problems – focus on calculus, not modeling.

Day 15: Intermediate Value Theorem





Focus on showing that the graphs of two functions intersect.

Day 16: Asymptotes and Limits to Infinity

From Stewart: "To indicate this kind of behavior we use the notation

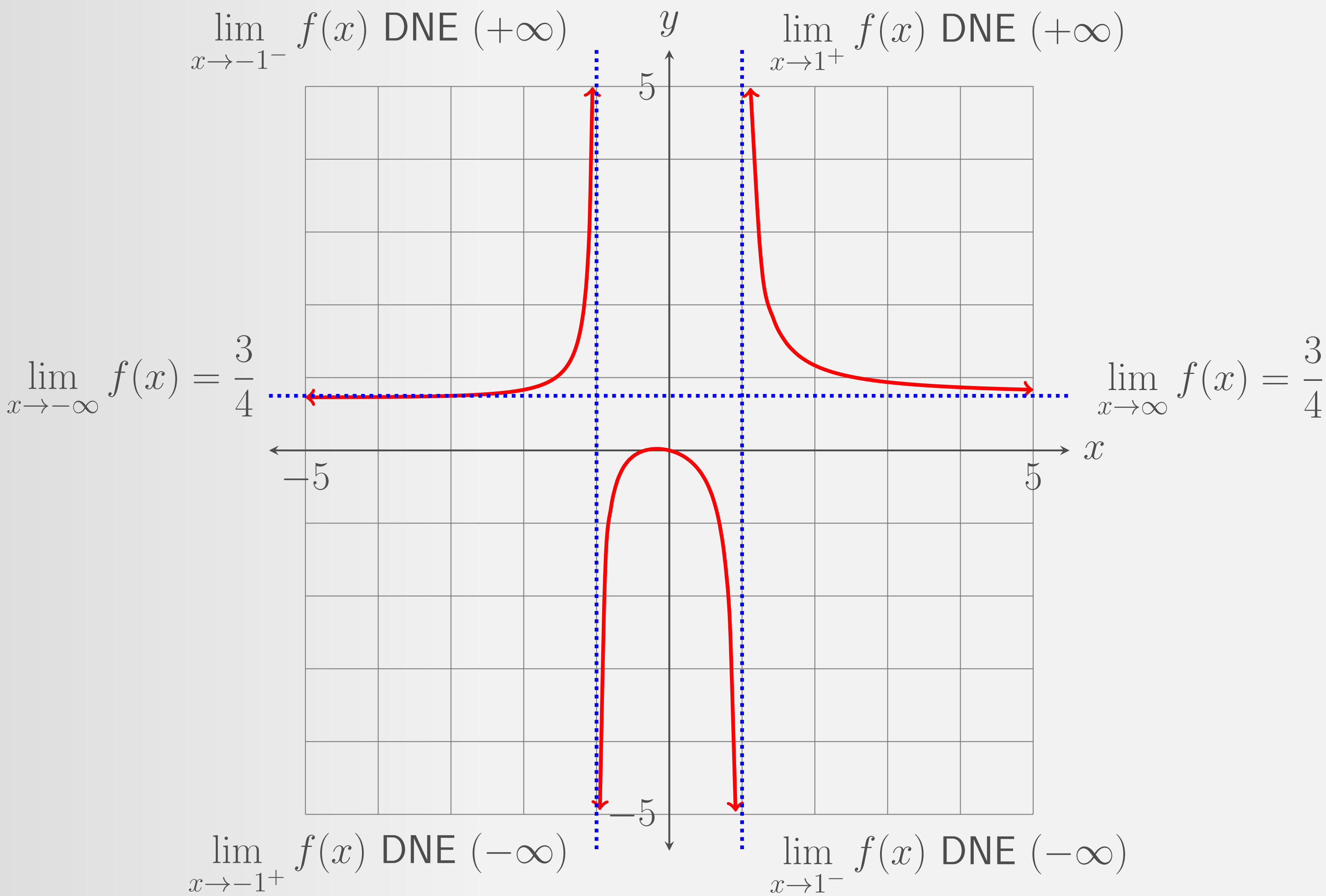
$$\lim_{x \to 0} \frac{1}{x^2} = \infty.$$

This does not mean that we are regarding ∞ as a number. Nor does it mean that the limit exists." !!!

Rather:

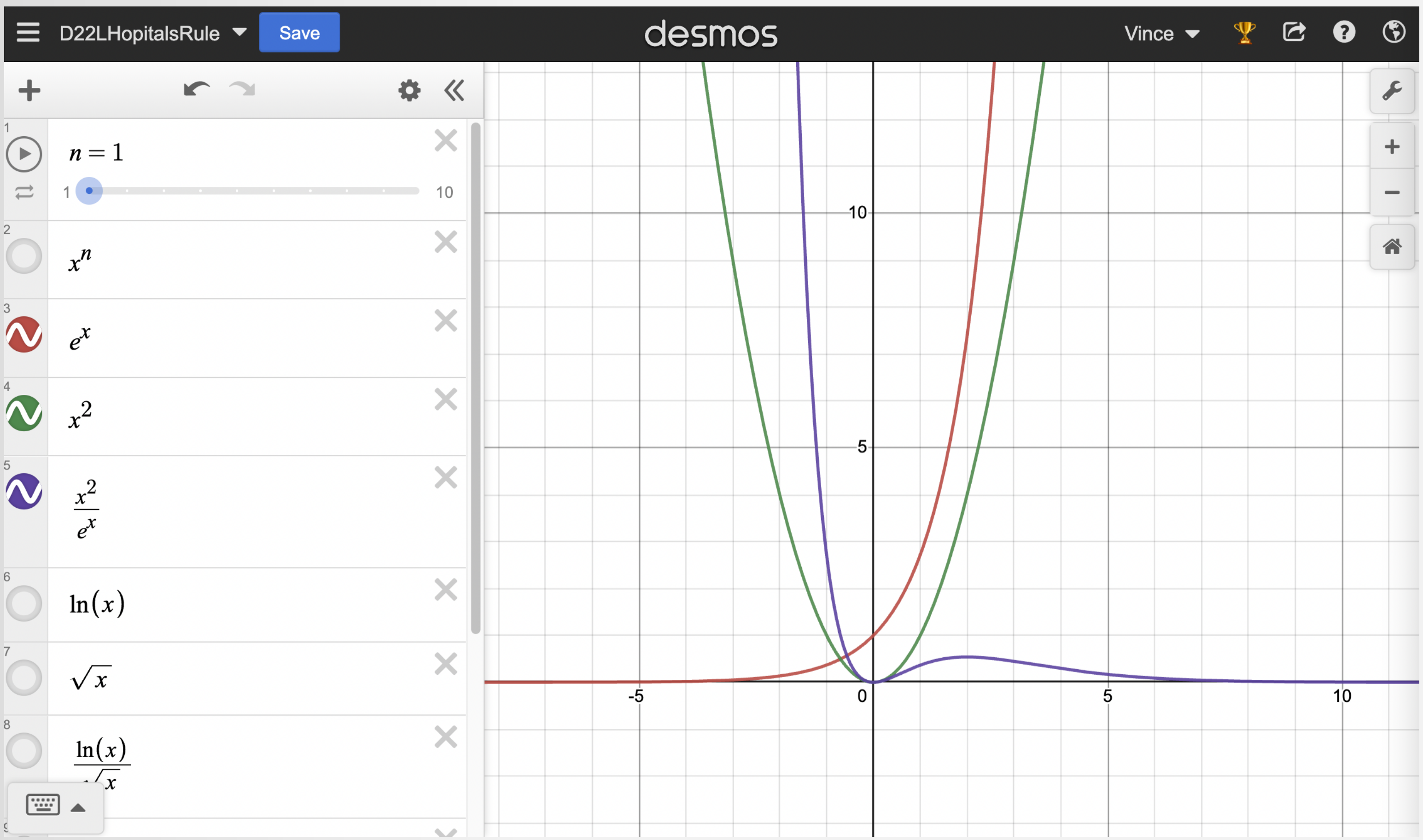
$$\lim_{x \to 0} \frac{1}{x^2} \text{ DNE } (+\infty).$$

Day 16: Asymptotes and Limits to Infinity



Day 17: L'Hôpital's Rule

Motivate L'Hôpital's Rule with desmos. Compare growth of functions by looking at their quotient.



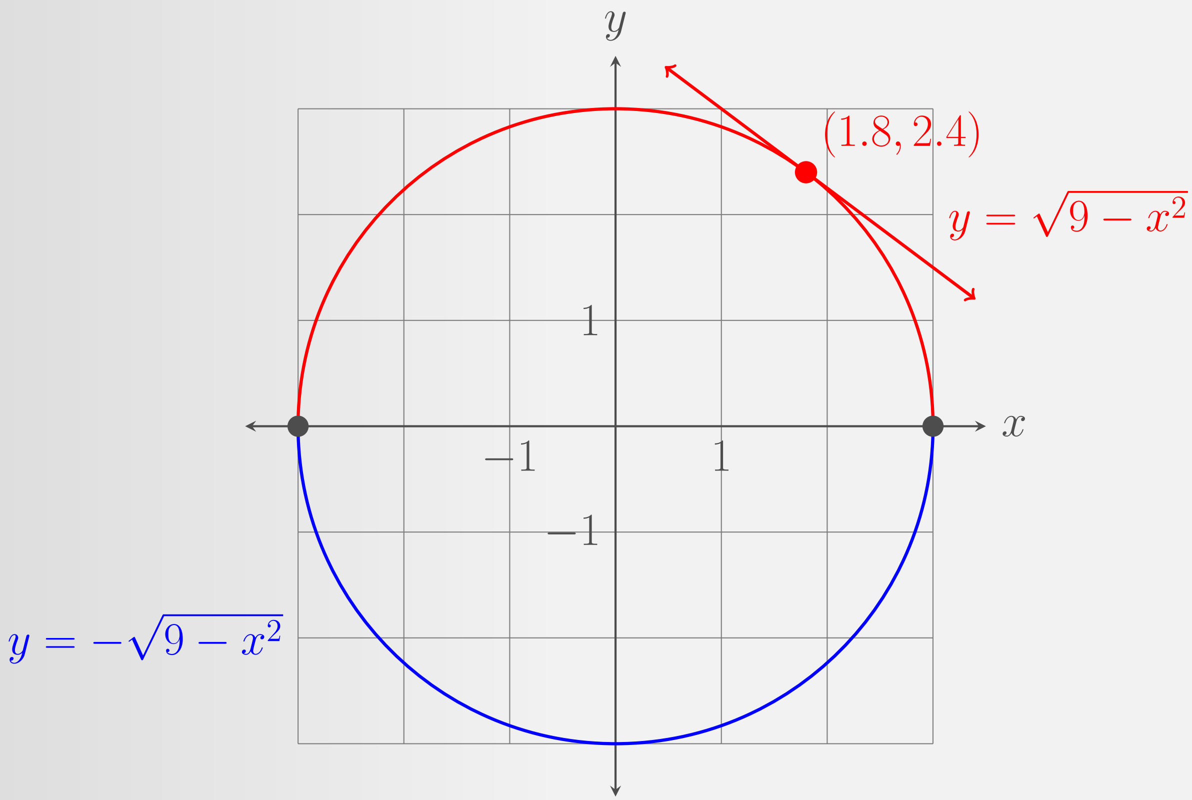
Day 18: More L'Hôpital's Rule

- Extend previous section by finding derivatives of b^x and $\log_b x$. Wait until they have facility with e^x and $\ln x$, since b^x and $\log_b x$ must be rewritten in terms of these.

- Discuss limits of the form $0 \cdot \infty$.

- Omit indeterminate limits of the form 0^0, ∞^0, and 1^∞. This would involve too much time with algebraic manipulation without introducing a new concept.

Day 19: Tangents to Curves

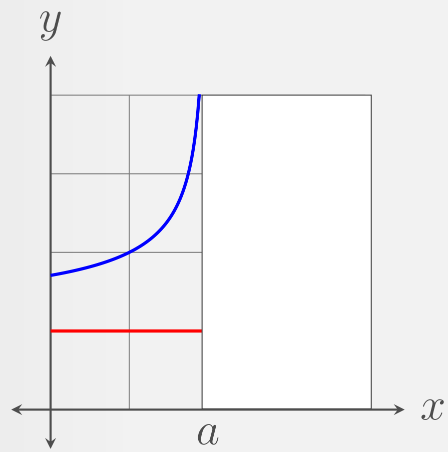


- Geometrical presentation.
- Emphasize ellipses and hyperbolas.
- Avoid complicated algebra.

Day 20: Summary of Limits in Calculus

The limit of $f(x)$ as x approaches a from the left:

$$\lim_{x \to a^-} f(x)$$



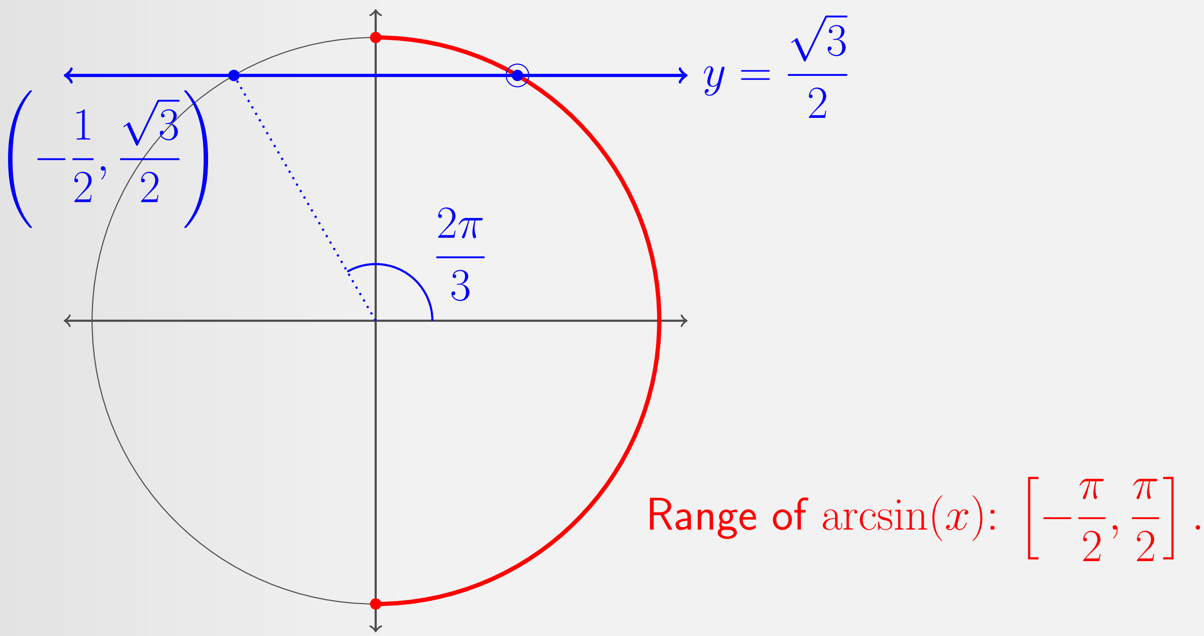
Used for:

1. Determining continuity,
2. Describing discontinuities,
3. Describing behavior at an asymptote.

As limits were gradually introduced when appropriate, here we summarize all the ways limits have been used.

Day 21: Inverse Trigonometric Functions

Finding $\arccos(\cos(2\pi/3))$.



This is a *very* abstract idea for many students.
Domain/range specifications were illustrated in desmos.
Derivatives were done the next class after a quiz.

Day 23: Calculus and Graphing

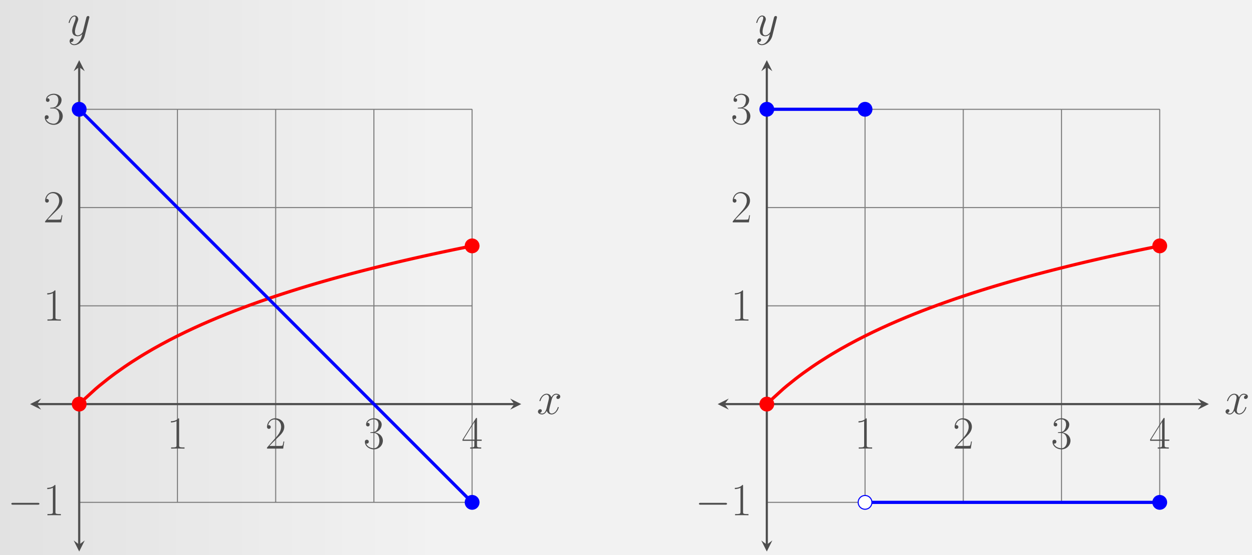
Given a function, its first two derivatives, and its graph:

- Determine horizontal and vertical asymptotes, if any;
- Determine local minima and maxima, if any;
- Determine intervals where the function is increasing and decreasing;
- Determine inflection points, if any;
- Determine intervals on which the graph is concave up or concave down.

Move towards graphical analysis, and away from curve sketching.

Day 24: Summary of Continuity and Differentiation

Continuity and the Intermediate Value Theorem:



Uses of continuity and differentiation in calculus are summarized before moving on to antiderivatives.

Homework Quizzes

- Students can bring their notebooks.
- Questions are similar to homework problems – worked example effect.

Pros:

- Encourages students to do homework.
- Keeping a notebook is active learning.
- Gives insight into what students are missing.
- Takes little time to write.

Cons:

- Grading.

Exams

- Write *calculus* exams, not *algebra* exams.
- Avoid multistep problems.
- Give derivatives if assessing other concepts.

Consider $f(x) = \dfrac{x-1}{x^3}$ (the graph is given). Here are the derivatives:

$$f'(x) = \frac{3-2x}{x^4}, \quad f''(x) = \frac{6(x-2)}{x^5}.$$

1. Find the x-values and y-values of the local maxima and minima. Label them on the graph. You do *not* have to show they are extrema. Just label them on the graph.
2. By creating a sign chart, find the intervals where $f(x)$ is concave up and concave down.

Extra Credit Assignments

- Students write a one-page critique on a section of the textbook.
- Worth +0.25% points each at the end of the semester.
- Can do up to eight assignments.
- Gets students reading.
- Students often give valuable comments, which I use when revising.

Contact Information

This talk is available at

vincematsko.com > MAA.

The text is available at

vincematsko.com > calculus.

Email me at

vince.matsko@gmail.com.

This is an open-source project. Right now, I'm working with PDFs and desmos. Collaborators with knowledge of other platforms are welcome!