Using Differentiation Rules

When you are given a function $f(x)$ and need to take the derivative, which rule(s) should you use? It's not always obvious.

Example 1

Let $f(x) = \dfrac{x^4 - 3x^2 + 5x}{x}$. Since $f(x)$ is a fraction, your first thought might be to use the quotient rule. This isn't wrong, but it's way too much work. In this case, it's easier to divide out the x first.

$$\begin{aligned} f(x) &= \frac{x^4 - 3x^2 + 5x}{x} \\ &= \frac{x^4}{x} - \frac{3x^2}{x} + \frac{5x}{x} \\ &= x^3 - 3x + 5 \end{aligned}$$

Now, it's easy to find $f'(x)$ using the power rule: $f'(x) = 3x^2 - 3$.

Example 2

Let $f(x) = \dfrac{\cos(x)}{x^{-2}}$. Again, it's tempting to use the quotient rule. But recall that

$$\frac{1}{x^{-2}} = x^2.$$

So it's easier to write $f(x) = x^2 \cos(x)$ and use the product rule.

$$\begin{aligned} f(x) &= x^2 \cos(x) \\ f'(x) &= (x^2)(-\sin(x)) + \cos(x)(2x) \\ &= -x^2 \sin(x) + 2x \cos(x). \end{aligned}$$

Example 3

Let $f(x) = (x^2 + 1)(x - 3)$. Yes, you can use the product rule here. But in this case, it's simpler to FOIL out $f(x)$ and then just use the power rule.

$$\begin{aligned} f(x) &= (x^2 + 1)(x - 3) \\ &= x^3 - 3x^2 + x - 3 \\ f'(x) &= 3x^2 - 6x + 1 \end{aligned}$$

The common theme here is that we rewrote each function so that an easier differentiation rule can be used. There's no "magic formula" for how to do this, you just have to practice.

But before jumping into a problem, it's always a good idea to take a moment to see if the function can be rewritten to make it easier to differentiate.

Example 4

Usually, when you see a function inside of another function, you need to use the chain rule. Occasionally, there may be a different way. Suppose $h(x) = (x^3+1)^2$. You might try the chain rule with $f(x) = x^2$ and $g(x) = x^3 + 1$. Then $f'(x) = 2x$ and $g'(x) = 3x^2$, so that

$$\begin{aligned} h'(x) &= f'(g(x))g'(x) \\ &= 2(g(x)) \cdot 3x^2 \\ &= 2(x^3+1)(3x^2) \\ &= 6x^2(x^3+1) \end{aligned}$$

But it also possible to FOIL out $f(x)$ first. Here's what you get.

$$\begin{aligned} h(x) &= (x^3+1)^2 \\ &= x^6 + 2x^3 + 1 \\ h'(x) &= 6x^5 + 6x^2 \end{aligned}$$

Since $6x^2(x^3+1) = 6x^5+6x^2$, both methods give the same answer. One way isn't necessarily easier than the other, so either way you choose to do it is OK.

Example 5

Let $f(x) = \dfrac{5}{x^6}$. This is a fraction, so you might be tempted to use the quotient rule. But it's easier to use rules of exponents to rewrite $f(x) = 5x^{-6}$. You cannot just use the power rule on the denominator; the *entire* function must be of the form ax^n for some n.

$$\begin{aligned} f(x) &= 5x^{-6} \\ f'(x) &= 5 \cdot (-6)x^{-6-1} \\ &= -30x^{-7} \end{aligned}$$

Note that the exponent must be "-7," not "-5," since we have to subtract 1 from the exponent.

Find the derivatives of the following functions. Some problems will be easier if you rewrite them first, so take a moment to look before you leap. Use Wolfram Alpha to check your answers.

1. $h(x) = \dfrac{2x}{3x^4}$
2. $h(x) = \dfrac{x^2+1}{x}$
3. $h(x) = \dfrac{x}{x^2+1}$
4. $h(x) = (5x-3)^{10}$
5. $h(x) = (5x-3)^{-10}$
6. $h(x) = x^3\sqrt{x}$
7. $h(x) = \sin(x)\sqrt{x}$
8. $h(x) = \dfrac{ax+b}{ax-b}$
9. $h(x) = \sin(x)\cos(x)$
10. $h(x) = \sqrt{3x-5}$
11. $h(x) = \sqrt[5]{1+\cos(x)}$
12. $h(x) = \cos(x^3)$
13. $h(x) = \cos^3(x)$
14. $h(x) = \tan(x)$ (Hint: Use the quotient rule.)
15. $h(x) = \dfrac{\sin(x)}{x^{-3}}$