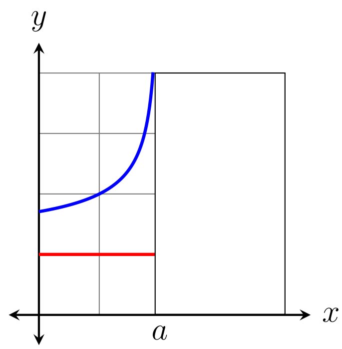
Summary of Limits in Calculus

We used limits at various different points so far. They were necessary to define the derivative. We also used them to define what it means to be continuous or discontinuous, which includes essential and removable discontinuites. Then we used limits to describe the behavior of the graph of a function at horizontal and vertical asymptotes. Here, we summarize all the important points.

The limit of $f(x)$ as x approaches a from the left:

$$\lim_{x\to a^-} f(x)$$

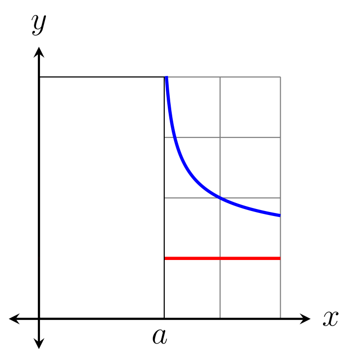


Used for:

1. Determining continuity,
2. Describing discontinuities,
3. Describing behavior at an asymptote.

The limit of $f(x)$ as x approaches a from the right:

$$\lim_{x\to a^+} f(x)$$



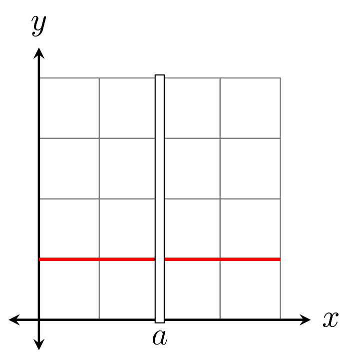
Used for:

1. Determining continuity,
2. Describing discontinuities,
3. Describing behavior at an asymptote.

The limit of $f(x)$ as x approaches a : $\lim_{x\to a} f(x)$.

Only exists if

$$\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x).$$



Used for:

1. Determining continuity,
2. Describing discontinuities.

The limit of $f(x)$ as x approaches infinity:

$$\lim_{x\to\infty} f(x)$$

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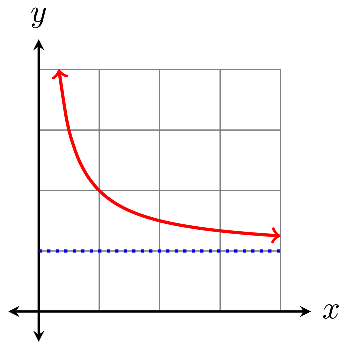

Used for:

1. Determining a horizontal asymptote to the right.

The limit of $f(x)$ as x approaches negative infinity:

$$\lim_{x\to-\infty} f(x)$$

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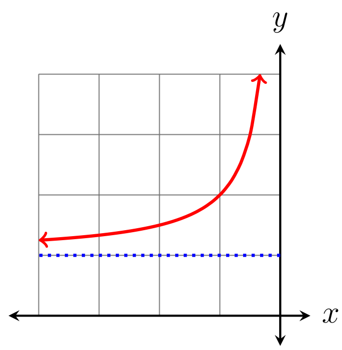
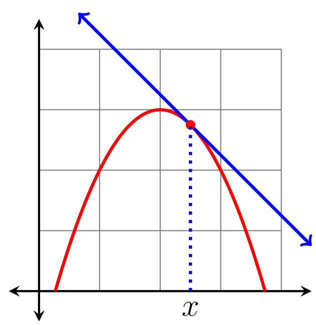


Used for:

1. Determining a horizontal asymptote to the left.

The derivative of $f(x)$:

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$



Used for:

1. Determining the slope of the tangent line.

Note that the h *must* cancel somehow.

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Evaluating Limits

There are many ways to evaluate limits. Let's summarize the ones we encounter most often.

In ALL cases where the limit DNE (does not exist), you must do additional work to see if the limit DNE ($-\infty$), DNE ($+\infty$), or just DNE (for example, when a function approaches an asymptote in opposite directions).

1. Sometimes you can just plug in. The commonly occurs when using the definition of the derivative. You can't plug in right away, since you get $\frac{0}{0}$. But once the h cancels out, you can usually just plug in.

2. When the limit involves a quotient, there are two primary methods.

   (a) If the limit is the limit of $x \to \pm\infty$ of a rational function $f(x)$ (numerator and denominator are polynomials), and if N is the degree of the numerator and D is the degree of the denominator, then:

      i. If $N < D$, then $\lim\limits_{x\to-\infty} f(x) = \lim\limits_{x\to\infty} f(x) = 0$;

      ii. If $N = D$, then $\lim\limits_{x\to-\infty} f(x) = \lim\limits_{x\to\infty} f(x)$ is the ratio of the leading coefficients of the numerator and the denominator;

      iii. If $N > D$, then $\lim\limits_{x\to-\infty} f(x)$ and $\lim\limits_{x\to\infty} f(x)$ DNE.

   (b) If the quotient does *not* involve a rational function, then evaluate using the following chart, where LR stands for L'Hôpital's Rule. Again, DNE can also mean DNE ($-\infty$) or DNE ($+\infty$), but more work usually has to be done to determine if one of these applies. Here, "$\neq 0$" means not 0, but *also* not $\pm\infty$.

DEN \ NUM	0	$\neq 0$	$\pm\infty$
0	LR	DNE	DNE
$\neq 0$	0	PLUG IN	DNE
$\pm\infty$	0	0	LR

3. When the limit involves a product, such as $\lim_{x\to a} f(x)g(x)$, where a can be a number or $\pm\infty$, you can often just plug in. If not, use the following chart. Here, the label "$f(x)$" means what $\lim_{x\to a} f(x)$ is, and "$g(x)$" means what $\lim_{x\to a} g(x)$ is. LR means that you have to rewrite the product as a quotient and use L'Hôpital's Rule. Two rules of thumb:

   (a) Move a term with a negative exponent to the denominator;

   (b) Leave a logarithm on the numerator.

   In all other cases, take your best guess. Remember, if you rewrite

$$\lim_{x\to a} f(x)g(x) = \lim_{x\to a} \frac{f(x)}{\dfrac{1}{g(x)}},$$

   you have to take the derivative of $\dfrac{1}{g(x)} = (g(x))^{-1}$ using the Chain Rule. Try to determine which way to rewrite which will be easiest.

$g(x)$ \ $f(x)$	0	$\neq 0$	$\pm\infty$
0	0	0	LR
$\neq 0$	0	PLUG IN	DNE
$\pm\infty$	LR	DNE	DNE

4. Some limits we just "know" and do not need to use the charts. They all just "make sense," or we studied them in detail in class. Here, "$n > 0$" means *any* positive number, not just an integer.

   (a) $\lim_{x\to\infty} x^n$ DNE $(+\infty)$, where $n > 0$.

   (b) $\lim_{x\to-\infty} x^n$ DNE, where $n > 0$ and x^n is well-defined.

   (c) $\lim_{x\to\infty} \frac{1}{x^n} = 0$, where $n > 0$.

   (d) $\lim_{x\to-\infty} \frac{1}{x^n} = 0$, where $n > 0$ and x^n is well-defined.

   (e) $\lim_{x\to 0^+} \frac{1}{x^n}$ DNE $(+\infty)$, where $n > 0$.

   (f) $\lim_{x\to 0^-} \frac{1}{x^n}$ DNE, where $n > 0$ and x^n is well-defined.

   (g) $\lim_{x\to\infty} b^x$ DNE $(+\infty)$, where $b > 1$.

   (h) $\lim_{x\to-\infty} b^x = 0$, where $b > 1$.

   (i) $\lim_{x\to\infty} \ln x$ DNE $(+\infty)$.

   (j) $\lim_{x\to 0^+} \ln x$ DNE $(-\infty)$.

   (k) $\lim_{x\to\infty} \frac{x^n}{b^x} = 0$, where $n > 0$ and $b > 1$.

   (l) $\lim_{x\to\infty} \frac{\log_b x}{x^n} = 0$, where $n > 0$ and $b > 1$.

In-Class Practice/Homework

When a limit DNE, determine whether it is DNE ($+\infty$), DNE ($-\infty$), or DNE.

1. $\displaystyle\lim_{x\to-\infty} \frac{1-2x^2}{3x^2-4x-1}$
2. $\displaystyle\lim_{x\to2^-} \frac{x^2+1}{x^2-4}$
3. $\displaystyle\lim_{x\to2^+} \frac{x^2+1}{x^2-4}$
4. $\displaystyle\lim_{x\to2} \frac{x^2+1}{x^2-4}$
5. $\displaystyle\lim_{x\to-\infty} \frac{x^4-2x-6}{x-4}$
6. $\displaystyle\lim_{x\to\infty} e^{-x}\ln x$
7. $\displaystyle\lim_{x\to0^+} e^{x}\sin(x)$
8. $\displaystyle\lim_{x\to0^-} \frac{x^2}{\sin(x)}$
9. $\displaystyle\lim_{x\to0^-} \frac{x^2+1}{\sin(x)}$
10. $\displaystyle\lim_{x\to-\infty} x^5\ln x$
11. $\displaystyle\lim_{x\to\infty} x^5\ln x$
12. $\displaystyle\lim_{x\to0^+} x^5\ln x$
13. $\displaystyle\lim_{x\to\infty} \frac{e^x}{x}$.
14. $\displaystyle\lim_{x\to\infty} \frac{2^x}{e^x}$.
15. $\displaystyle\lim_{x\to0^-} \frac{\sin(x)}{\tan(x)}$.

Solutions

1. This is a rational function with $N = 2$ and $D = 2$. Therefore, the limit is the ratio of the leading coefficients, $-\dfrac{2}{3}$.

2. The numerator approaches 5, while the denominator approaches 0. Thus, this limit DNE. Since the numerator is positive and the denominator is negative – plug in 1.99 to get $1.99^2 - 4 \approx -0.04$ – the limit is DNE $(-\infty)$.

3. The numerator approaches 5, while the denominator approaches 0. Thus, this limit DNE. Since the numerator is positive and the denominator is positive – plug in 2.01 to get $2.01^2 - 4 \approx 0.04$ – the limit is DNE $(+\infty)$.

4. Based on the previous two answers, this limit is DNE, since the graph approaches the asymptote $x = 2$ from opposite directions.

5. $N = 4$ and $D = 1$, so since $N > D$, this limit DNE. To see if it is DNE $(+\infty)$ or DNE $(-\infty)$, we look at the highest degrees of the numerator and denominator here as $x \to -\infty$. So
$$\frac{x^4 - 2x - 6}{x - 4} \approx \frac{x^4}{x} = x^3.$$
Since the cube of a negative number is negative, this limit is DNE $(-\infty)$.

6. This limit is of the form $0 \cdot \infty$. We move the negative exponent to the denominator and use L'Hôpital's Rule.
$$\lim_{x\to\infty} e^{-x} \ln x = \lim_{x\to\infty} \frac{\ln x}{e^x} \stackrel{\text{LR}}{=} \lim_{x\to\infty} \frac{\frac{1}{x}}{e^x} = \lim_{x\to\infty} \frac{1}{xe^x} = 0.$$

7. Here, we can just plug in. $\lim\limits_{x\to 0^+} e^x \sin(x) = e^0 \sin(0) = 0$.

8. This limit is of the form $\dfrac{0}{0}$, so we can apply L'Hôpital's Rule.
$$\lim_{x\to 0^-} \frac{x^2}{\sin(x)} \stackrel{\text{LR}}{=} \lim_{x\to 0^-} \frac{2x}{\cos(x)} = \frac{2 \cdot 0}{\cos(0)} = 0.$$

9. The numerator goes to 1 and the denominator goes to 0, so this limit DNE. To see if it is DNE $(+\infty)$ or DNE $(-\infty)$, note that 1 is positive. Looking at a graph of $y = \sin(x)$, we see that coming to 0 from the left, $\sin(x)$ is negative. Since $\dfrac{+}{-} = -$, this limit is DNE $(-\infty)$.

10. This limit is undefined, since $\ln x$ is not defined for negative numbers. It is important to point out that "undefined" is different that "DNE." To say that a limit is undefined is to say that you can't even evaluate it because the x-values don't make sense for the functions in the limit. To say that a limit is DNE means the limit makes sense as far as the x-values are concerned, but there is no limiting value.

11. This limit is of the form $+\infty \cdot +\infty$, so this limit DNE $(+\infty)$.

12. This limit is of the form $0 \cdot -\infty$, so we must rewrite and use L'Hôpital's Rule. We keep the logarithm on the numerator.

$$\lim_{x\to 0^+} x^5 \ln x = \lim_{x\to 0^+} \frac{\ln x}{x^{-5}} \overset{\text{LR}}{=} \lim_{x\to 0^+} \frac{\frac{1}{x}}{-5x^{-6}} \cdot \frac{x^6}{x^6} = \lim_{x\to 0^+} \frac{x^5}{-5} = 0.$$

13. We know that this limit DNE since e^x dominates x. As $x \to \infty$, this limit is of the form $\frac{+}{+} = +$, so the limit is DNE $(+\infty)$.

14. This limit is of the form $\frac{\infty}{\infty}$, but using L'Hôpital's Rule won't help because the exponential functions will still remain. But we can rewrite as $\lim_{x\to\infty} \left(\frac{2}{e}\right)^x = 0$. Since $2 < e$, we are taking higher and higher powers of a number less than 1, so this limit is 0.

15. This limit is of the form $\frac{0}{0}$, so it looks like a L'Hôpital's Rule problem. But

$$\frac{\sin(x)}{\tan(x)} = \frac{\sin(x)}{\frac{\sin(x)}{\cos(x)}} = \frac{\sin(x)}{1} \cdot \frac{\cos(x)}{\sin(x)} = \cos(x),$$

so this limit is

$$\lim_{x\to 0^-} \cos(x) = 1.$$