The general form of a linear transformation is

$$A\begin{pmatrix} x \\ y \end{pmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{pmatrix} x \\ y \end{pmatrix}.$$

Note that

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ c \end{pmatrix}, \qquad \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} b \\ d \end{pmatrix},$$

so the columns of the matrix are determined by the action of the linear transformation on the unit basis vectors.

The general form of an affine transformation is

$$A\begin{pmatrix} x \\ y \end{pmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} e \\ f \end{pmatrix},$$

where $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$ is the *linear part* and $\begin{pmatrix} e \\ f \end{pmatrix}$ is the *translation.*

Recall that

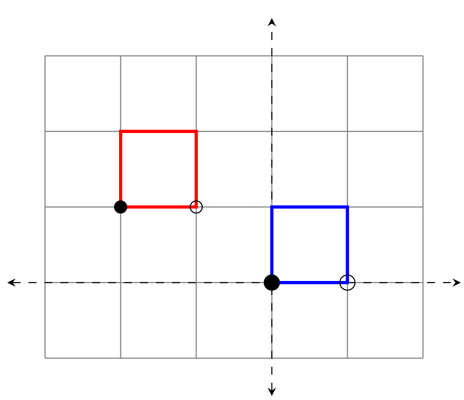
$$\det A = ad - bc$$

is called the *determinant* of the transformation. A reflection is involved whenever $\det A < 0$.

In the following library of affine transformations, the solid dot indicates where the origin is transformed to, and the open dot indicates where $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is transformed to. When a reflection is involved, the transformed parallelogram is shaded in light red. This is just a visual aid, since when a reflection is involved, moving from the solid dot to the open dot will involve a clockwise movement around the red parallelogram (as opposed to a counterclockwise movement around the blue unit square).

1. Identity transformation. Here, the unit square remains exactly the same but is translated.

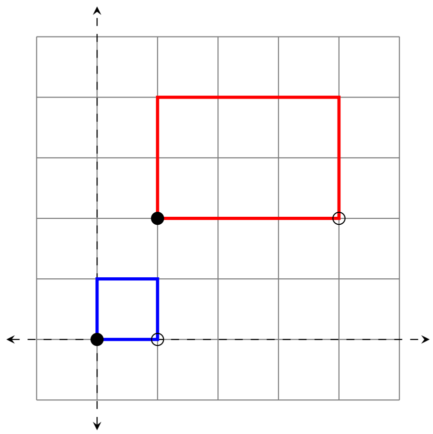
$$A\begin{pmatrix} x \\ y \end{pmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

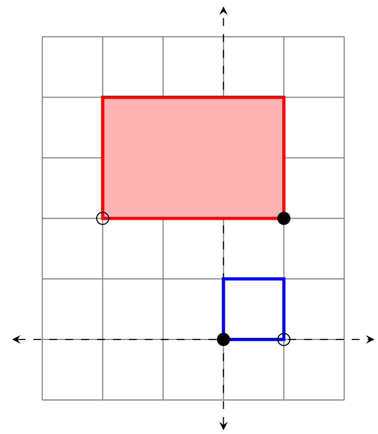


2. Horizontal and vertical scalings. Note that if a scale factor is negative, a reflection is involved.

$$A\begin{pmatrix}x\\y\end{pmatrix}=\begin{bmatrix}3&0\\0&2\end{bmatrix}\begin{pmatrix}x\\y\end{pmatrix}+\begin{pmatrix}1\\2\end{pmatrix}$$

$$A\begin{pmatrix}x\\y\end{pmatrix}=\begin{bmatrix}-3&0\\0&2\end{bmatrix}\begin{pmatrix}x\\y\end{pmatrix}+\begin{pmatrix}1\\2\end{pmatrix}$$

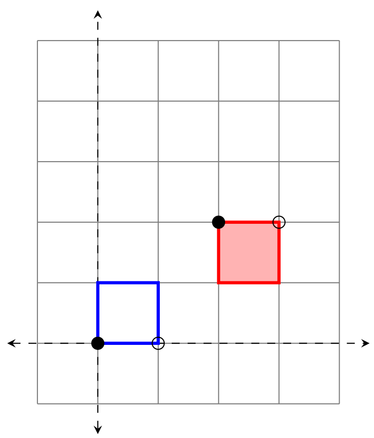


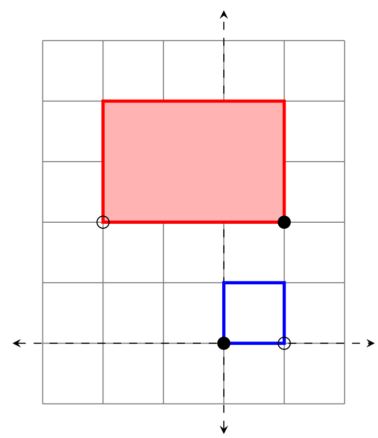


3. Reflections about the coordinate axes. On the left is a reflection in the x-axis. On the right is a reflection about the y-axis – note that scale factors are often included in reflections. This is equivalent to using a negative scale factor (see the previous example).

$$A\begin{pmatrix}x\\y\end{pmatrix}=\begin{bmatrix}1&0\\0&-1\end{bmatrix}\begin{pmatrix}x\\y\end{pmatrix}+\begin{pmatrix}2\\2\end{pmatrix}$$

$$A\begin{pmatrix}x\\y\end{pmatrix}=\begin{bmatrix}-3&0\\0&2\end{bmatrix}\begin{pmatrix}x\\y\end{pmatrix}+\begin{pmatrix}1\\2\end{pmatrix}$$

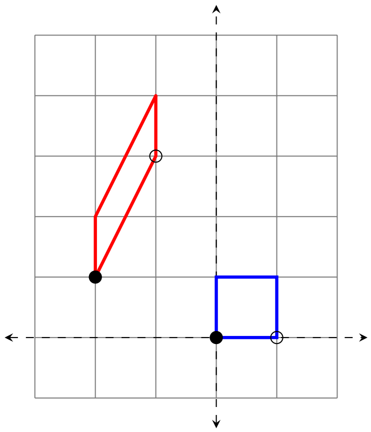


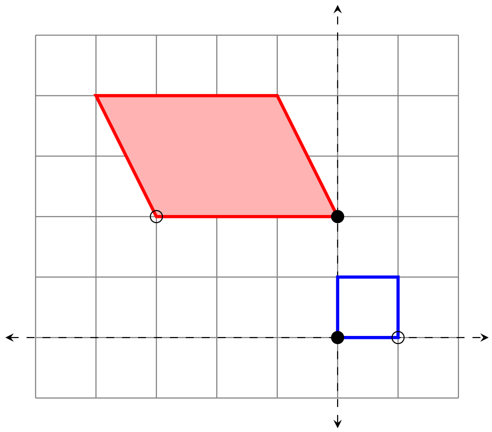


4. Horizontal and vertical shears. An example of a vertical shear is shown on the left. A horizontal shear is shown on the right – note that including a scale factor (such as scaling the x by -3) may also be involved.

$$A\begin{pmatrix}x\\y\end{pmatrix}=\begin{bmatrix}1&0\\2&1\end{bmatrix}\begin{pmatrix}x\\y\end{pmatrix}+\begin{pmatrix}-2\\1\end{pmatrix}$$

$$A\begin{pmatrix}x\\y\end{pmatrix}=\begin{bmatrix}-3&-1\\0&2\end{bmatrix}\begin{pmatrix}x\\y\end{pmatrix}+\begin{pmatrix}0\\2\end{pmatrix}$$

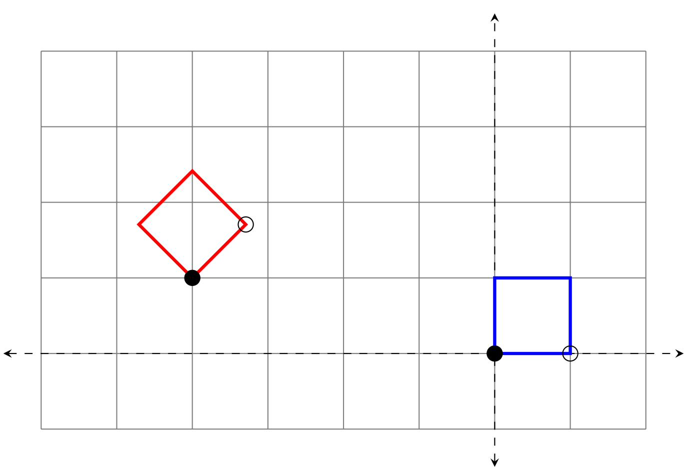




5. Rotations. The general form for a counterclockwise rotation about an angle θ is given by

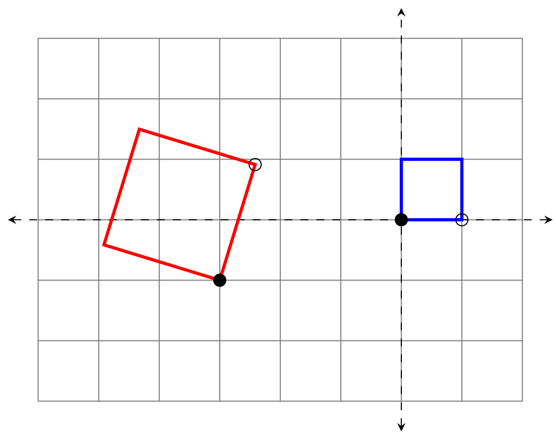
$$\mathbf{R}_\theta=\begin{bmatrix}\cos\theta&-\sin\theta\\\sin\theta&\cos\theta\end{bmatrix}.$$

$$A\begin{pmatrix}x\\y\end{pmatrix}=\mathbf{R}_{45^\circ}\begin{pmatrix}x\\y\end{pmatrix}+\begin{pmatrix}-4\\1\end{pmatrix}=\begin{bmatrix}0.707&-0.707\\0.707&0.707\end{bmatrix}\begin{pmatrix}x\\y\end{pmatrix}+\begin{pmatrix}-4\\1\end{pmatrix}$$



A scale factor is often incorporated into the rotation matrix.

$$A\begin{pmatrix}x\\y\end{pmatrix} = 2\cdot\mathbf{R}_{73^\circ} + \begin{pmatrix}-3\\-1\end{pmatrix} = \begin{bmatrix}0.585 & -1.913\\1.913 & 0.585\end{bmatrix}\begin{pmatrix}x\\y\end{pmatrix} + \begin{pmatrix}-3\\-1\end{pmatrix}$$

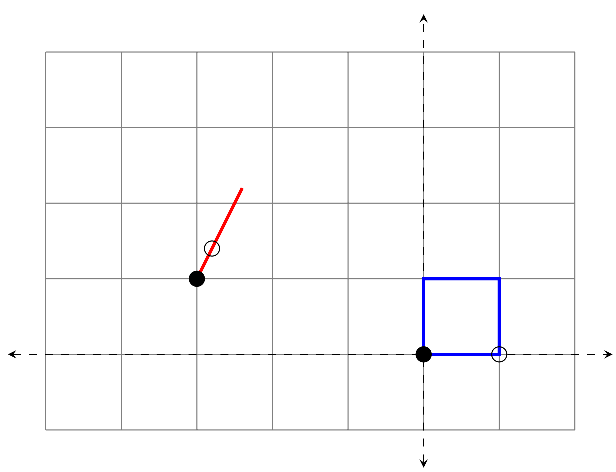


6. Projection onto the line $y = mx$. The general form for this matrix is

$$\mathbf{P}_m = \begin{bmatrix}\dfrac{1}{m^2+1} & \dfrac{m}{m^2+1}\\ \dfrac{m}{m^2+1} & \dfrac{m^2}{m^2+1}\end{bmatrix}.$$

Note that a projection will always transform the unit square into a line segment; in other words, $\det\mathbf{P}_m$ is always 0.

$$A\begin{pmatrix}x\\y\end{pmatrix} = \mathbf{P}_2\begin{pmatrix}x\\y\end{pmatrix} + \begin{pmatrix}-3\\1\end{pmatrix} = \begin{bmatrix}1/5 & 2/5\\2/5 & 4/5\end{bmatrix}\begin{pmatrix}x\\y\end{pmatrix} + \begin{pmatrix}-3\\1\end{pmatrix}$$

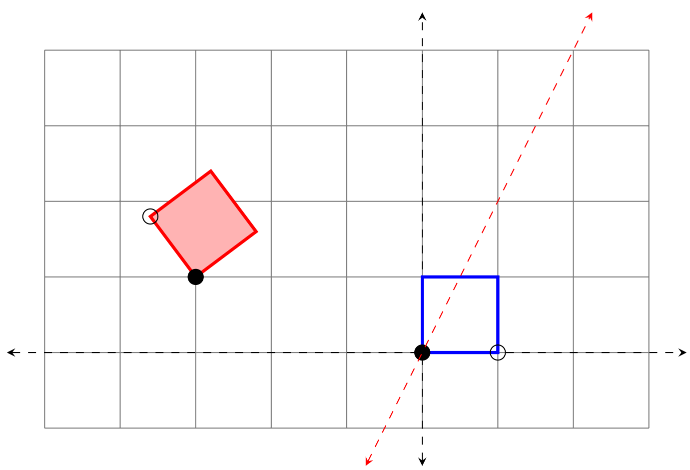


7. Reflection about the line $y = mx$. The general form for this matrix is

$$\mathbf{S}_m = \begin{bmatrix} \dfrac{1-m^2}{m^2+1} & \dfrac{2m}{m^2+1} \\ \dfrac{2m}{m^2+1} & \dfrac{m^2-1}{m^2+1} \end{bmatrix}.$$

It is not hard to work out that $\det \mathbf{S}_m$ is always -1.

$$A\begin{pmatrix} x \\ y \end{pmatrix} = \mathbf{S}_2 \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} -3 \\ 1 \end{pmatrix} = \begin{bmatrix} -3/5 & 4/5 \\ 4/5 & 3/5 \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} -3 \\ 1 \end{pmatrix}$$



8. Composition of transformations. It is not difficult to combine different transformations. For example, if you wanted to first perform a vertical shear (Example 4) and then rotate counterclockwise through 45° (Example 5), you would use matrix multipliation (that is, function composition) to get the linear part, as shown below. Note the order in which you write the matrices!

$$\begin{bmatrix} 0.707 & -0.707 \\ 0.707 & 0.707 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} -0.707 & -0.707 \\ 2.121 & 0.707 \end{bmatrix}.$$