1. Circle TRUE or FALSE for each question.

   (a) TRUE FALSE $\lambda = 1$ is an eigenvalue for the reflection across the line $y = -2x$.

   (b) TRUE FALSE $\lambda = -1$ is an eigenalue for the projection onto the line $y = -2x$.

   (c) TRUE FALSE If $0° < D < 360°$, then a rotation in the xy-plane through D has no real eigenvalues.

   (d) TRUE FALSE Any elementary matrix is also a symmetric matrix.

   (e) TRUE FALSE The geometric multiplicity and the algebraic multiplicity of an eigenvalue are always equal.

2. Suppose a symmetric matrix with distinct eigenvalues has $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$ as an eigenvector. Then another eigenvector for this matrix is:

   (a) $\begin{pmatrix} -3 \\ 1 \end{pmatrix}$ (b) $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$ (c) $\begin{pmatrix} -1 \\ 3 \end{pmatrix}$ (d) $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ (e) None of these.

3. Which of the following are eigenvectors of $\begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$? (There may be more than one answer.)

   (a) $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ (b) $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ (c) $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ (d) $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ (e) None of these.

4. $p_{12}p_{13}$ is equal to:

   (a) p_{12} (b) p_{13} (c) p_{23} (d) I (e) None of these.

5. Consider the recurrence $a_{n+2} = 2a_{n+1} - 3a_n$, $a_0 = 1$, $a_1 = 3$. Which of the following matrices would you use to solve this recurrence with matrix methods?

   (a) $\begin{bmatrix} 0 & 1 \\ 2 & -3 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 0 \\ 2 & -3 \end{bmatrix}$ (c) $\begin{bmatrix} 0 & 1 \\ -3 & 2 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 0 \\ -3 & 2 \end{bmatrix}$ (e) None of these.

6. Find the inverse of the matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

7. You know that a recurrence with $a_0 = 0$ and $a_1 = 1$ has the form $a_n = c_1 \cdot 8^n + c_2 \cdot 4^n$. Find c_1 and c_2.

$a_0 = C_1 + C_2 = 0 \Rightarrow C_2 = -C_1$

$a_1 = 8C_1 + 4C_2 = 1$

$8C_1 - 4C_1 = 1$

$4C_1 = 1 \Rightarrow C_1 = \frac{1}{4}, \; C_2 = -\frac{1}{4}$

8. Find all c, d for which the following system has infinitely many solutions.

$$3x - y = c$$
$$-6x + dy = 12$$

$\times -2$

$c = -6$

$d = 2$

9. Find the eigenvalues of $\begin{bmatrix} 4 & 1 \\ 7 & -2 \end{bmatrix}$.

$\det \begin{bmatrix} 4-\lambda & 1 \\ 7 & -2-\lambda \end{bmatrix} = (4-\lambda)(-2-\lambda) - 7 = 0$

$\lambda^2 - 2\lambda - 15 = 0$

$(\lambda - 5)(\lambda + 3) = 0$

$\lambda = 5, \; \lambda = -3$

10. Find the inverse of the matrix $\begin{bmatrix} 6 & -5 \\ -3 & 2 \end{bmatrix}$. Simplify as much as possible by multiplying out any fractions.

determinant $= 6 \cdot 2 - (-3)(-5) = -3$

$-\frac{1}{3} \begin{bmatrix} 2 & 5 \\ 3 & 6 \end{bmatrix} = \begin{bmatrix} -2/3 & -5/3 \\ -1 & -2 \end{bmatrix}$

11. Suppose you know that

$$A = LDU = \begin{bmatrix} 1 & 0 \\ a & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & b \end{bmatrix} \begin{bmatrix} 1 & c \\ 0 & 1 \end{bmatrix}$$

and that $\det A = 10$. Which of a, b, and/or c can you determine? Find the values of those you can determine.

You can only determine b: $2b = 10$

$b = 5$

12. The matrix $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ has eigenvalues of $\lambda = 3$ and $\lambda = 1$. The corresponding eigenvectors are $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$. Use this to write $A = PDP^{-1}$ and calculate A^3.

$P = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ $D = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$ $P^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$

$$A^3 = PD^3P^{-1} = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}^3\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 27 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 27 & 27 \\ -1 & 1 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 28 & 26 \\ 26 & 28 \end{bmatrix}$$

$$= \begin{bmatrix} 14 & 13 \\ 13 & 14 \end{bmatrix}$$

13. You are solving the following system of equations:

$\frac{1}{2}(4) + a = 0 \Rightarrow a = -2$

$$4x - 3y + 7z = 10$$
$$ax - 4y + 3x = 5$$
$$bx + 5y + z = -9.$$

$-2(4) + b = 0 \Rightarrow b = 8$

Your first two steps involved elementary matrices $e_{21}^{1/2}$ and e_{31}^{-2}. What are a and b?

14. If $\begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 3 \\ -1 & 2 & 1 \end{bmatrix}$ has an eigvenvalue of $\lambda = -2$, find a corresponding eigenvector.

$$\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

$$\begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 3 \\ -1 & 2 & 1 \end{bmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = -2 \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\begin{aligned} y - z &= -2x \\ -x \quad + 3z &= -2y \\ -x + 2y + z &= -2z \end{aligned} \rightarrow \begin{aligned} 2x + y - z &= 0 \\ -x + 2y + 3z &= 0 \\ -x + 2y + 3z &= 0 \end{aligned}$$

$\times 2$

$5y + 5z = 0$

$y = -z$

Choose $z = 1$, $y = -1$,

Then $x = 1$.

15. What is your favorite color?

Extra Credit: Let E be a linear transformation such that $E^2 = E$. What are the possible eigenvalues for E?

Suppose $E\underline{v} = \lambda \underline{v}$. Then

$$E^2 \underline{v} = E(E\underline{v}) = E(\lambda \underline{v}) = \lambda E(\underline{v})$$

$$= \lambda(\lambda \underline{v}) = \lambda^2 \underline{v}.$$

But $E^2 \underline{v} = E\underline{v} = \lambda \underline{v}$, and so $\lambda^2 \underline{v} = \lambda \underline{v}$.

Since $\underline{v} \neq \underline{0}$, then $\lambda^2 = \lambda$, so λ is either

0 or 1.

1. Circle TRUE or FALSE for each question.

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   (a) $\begin{pmatrix}0\\0\end{pmatrix}$ (b) $\begin{pmatrix}-3\\1\end{pmatrix}$ (c) $\begin{pmatrix}1\\3\end{pmatrix}$ (d) $\begin{pmatrix}-1\\3\end{pmatrix}$ (e) None of these.

3. Which of the following are eigenvectors of $\begin{bmatrix}0 & -1 & 0\\1 & 0 & 0\\0 & 0 & 1\end{bmatrix}$? (There may be more than one answer.)

   (a) $\begin{pmatrix}1\\0\\0\end{pmatrix}$ (b) $\begin{pmatrix}0\\1\\0\end{pmatrix}$ (c) $\begin{pmatrix}0\\0\\1\end{pmatrix}$ (d) $\begin{pmatrix}1\\1\\1\end{pmatrix}$ (e) None of these.

4. $p_{23}p_{13}$ is equal to:

   (a) p_{12} (b) p_{13} (c) p_{23} (d) I (e) None of these.

5. Consider the recurrence $a_{n+2} = 3a_{n+1} - 4a_n$, $a_0 = 1$, $a_1 = 3$. Which of the following matrices would you use to solve this recurrence with matrix methods?

   (a) $\begin{bmatrix}0 & 1\\-4 & 3\end{bmatrix}$ (b) $\begin{bmatrix}1 & 0\\-4 & 3\end{bmatrix}$ (c) $\begin{bmatrix}0 & 1\\3 & -4\end{bmatrix}$ (d) $\begin{bmatrix}1 & 0\\3 & -4\end{bmatrix}$ (e) None of these.

6. Find the inverse of the matrix $\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 2 & 1\end{bmatrix}$.

$$\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & -2 & 1\end{bmatrix}$$

7. You know that a recurrence with $a_0 = 0$ and $a_1 = 1$ has the form $a_n = c_1 \cdot 7^n + c_2 \cdot 3^n$. Find c_1 and c_2.

$a_0 = c_1 + c_2 = 0 \Rightarrow c_2 = -c_1$

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$7c_1 - 3c_1 = 1$

$4c_1 = 1 \Rightarrow c_1 = \frac{1}{4}, \quad c_2 = \frac{-1}{4}$

8. Find all c, d for which the following system has infinitely many solutions.

$$\begin{aligned} 4x + y &= c \\ -8x + dy &= 14 \end{aligned}$$

$\times -2$

$c = -7$

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9. Find the eigenvalues of $\begin{bmatrix} 2 & 1 \\ 7 & -4 \end{bmatrix}$.

$\det \begin{bmatrix} 2-\lambda & 1 \\ 7 & -4-\lambda \end{bmatrix} = (2-\lambda)(-4-\lambda) - 7 = 0$

$\lambda^2 + 2\lambda - 15 = 0$

$(\lambda + 5)(\lambda - 3) = 0$

$\lambda = -5, \quad \lambda = 3$

10. Find the inverse of the matrix $\begin{bmatrix} 2 & -3 \\ -5 & 6 \end{bmatrix}$. Simplify as much as possible by multiplying out any fractions.

determinant $= 2 \cdot 6 - (-5)(-3) = -3$

$-\frac{1}{3} \begin{bmatrix} 6 & 3 \\ 5 & 2 \end{bmatrix} = \begin{bmatrix} -2 & -1 \\ -5/3 & -2/3 \end{bmatrix}$

11. Suppose you know that

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and that $\det A = 12$. Which of a, b, and/or c can you determine? Find the values of those you can determine.

You can only determine b: $3b = 12$
$b = 4$

12. The matrix $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ has eigenvalues of $\lambda = 3$ and $\lambda = -1$. The corresponding eigenvectors are $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$. Use this to write $A = PDP^{-1}$ and calculate A^3.

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$$\begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 3 \\ -1 & 2 & 1 \end{bmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = -2 \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$y - z = -2x$ $\qquad 2x + y - z = 0$

$-x + 3z = -2y \rightarrow -x + 2y + 3z = 0$ $\times 2$ $\qquad 5y + 5z = 0$

$-x + 2y + z = -2z$ $\qquad -x + 2y + 3z = 0$ $\qquad y = -z.$

Choose $z = 1$, $y = -1$.

Then $x = 1$

15. What is your favorite color?

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0 or 1