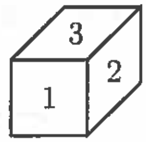
When appropriate, consider this the "start" position.



1. Give a parametric representation of the line through the points $\begin{pmatrix} 1 \\ -3 \end{pmatrix}$ and $\begin{pmatrix} 4 \\ 2 \end{pmatrix}$.

Vector between $\begin{pmatrix} 1 \\ -3 \end{pmatrix}$ and $\begin{pmatrix} 4 \\ 2 \end{pmatrix}$: $\begin{pmatrix} 4 \\ 2 \end{pmatrix} - \begin{pmatrix} 1 \\ -3 \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$

$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \end{pmatrix} + t\begin{pmatrix} 3 \\ 5 \end{pmatrix}$

$= \begin{pmatrix} 1+3t \\ -3+5t \end{pmatrix}$

$x = 1+3t$
$y = -3+5t$

Other possible answers:

$x = 4+3t$, $y = 2+5t$ $\quad$ $x = 1-3t$, $y = -3-5t$ $\quad$ $x = 4-3t$, $y = 2-5t$

2. Find all x such that $\det \begin{bmatrix} 5 & x \\ -3 & 4 \end{bmatrix} \leq 0$.

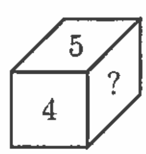
$\det \begin{bmatrix} 5 & x \\ -3 & 4 \end{bmatrix} = 5 \cdot 4 - (-3) \cdot x = 20 + 3x \leq 0$

$x \leq \frac{-20}{3}$

3. Find the following matrix product: $\begin{bmatrix} 4 & 3 \\ -1 & 5 \end{bmatrix} \cdot \begin{bmatrix} -2 & 1 \\ 3 & 1 \end{bmatrix}$.

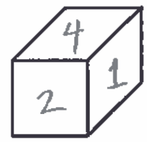
$\begin{bmatrix} 1 & 7 \\ 17 & 4 \end{bmatrix}$

4. Write the matrix which transforms the die to the following position:



$\begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}$

5. Fill in the die after performing the transformation $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$.



6. The matrix $\begin{bmatrix} 9/25 & -12/25 \\ -12/25 & 16/25 \end{bmatrix}$ is the matrix for the projection on a line L. Find a matrix for the reflection across the line L.

$$S = 2P - I = 2\begin{bmatrix} 9/25 & -12/25 \\ -12/25 & 16/25 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -7/25 & -24/25 \\ -24/25 & 7/25 \end{bmatrix}$$

7. How many direct symmetries of the cube are 90° rotations (either clockwise or counterclockwise) about an axis?

The only way to get a 90° rotation is by holding a die by opposite faces and turn 90° one way or the other.

So $3 \times 2 = 6$

8. Find symmetric equations of the line through the points $(3, 4, -2)$ and $(5, -1, 3)$.

First, find the direction of the line : $\begin{pmatrix} 5 \\ -1 \\ 3 \end{pmatrix} - \begin{pmatrix} 3 \\ 4 \\ -2 \end{pmatrix} = \begin{pmatrix} 2 \\ -5 \\ 5 \end{pmatrix}$

$$\frac{x-3}{2} = \frac{y-4}{-5} = \frac{z+2}{5} \quad \text{OR} \quad \frac{x-5}{2} = \frac{y+1}{-5} = \frac{z-3}{5}$$

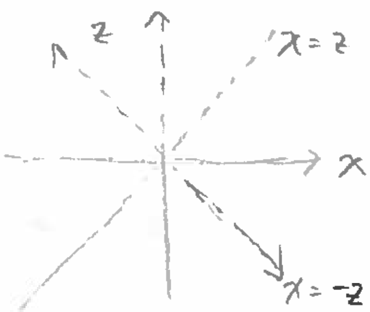
9. Find an equation for the plane with normal $(3, 2, -1)$ which passes through the point $(5, 1, 2)$.

$$3x + 2y - z = 3(5) + 2(1) - 1(2)$$
$$= 15$$

10. Find all eigenvalues and eigenvectors of $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$.

leaves y fixed: $\lambda = 1$, $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

Swaps x and z - reflection in the xz-plane about the line $x = z$



$\lambda = 1$, $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

$\lambda = -1$, $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$

11. The matrix $\begin{bmatrix} 4 & -3 \\ -1 & 2 \end{bmatrix}$ has an eigvenvalue of $\lambda = 5$. Find a corresponding eigenvector.

$$\begin{bmatrix} 4 & -3 \\ -1 & 2 \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 5 \begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow \begin{matrix} 4x - 3y = 5x \\ -x + 2y = 5y \end{matrix} \Rightarrow \begin{matrix} x = -3y \\ x = -3y \end{matrix}$$

eigenvectors parallel to $\begin{pmatrix} -3 \\ 1 \end{pmatrix}$

12. Find the inverse of the matrix $\begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

This is an elementary matrix.

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

13. Find the inverse of the matrix $\begin{bmatrix} 4 & -2 \\ 3 & 6 \end{bmatrix}$. Simplify as much as possible by multiplying out any fractions.

Determinant $= 4(6) - 3(-2) = 30$

Inverse : $\frac{1}{30}\begin{bmatrix} 6 & 2 \\ -3 & 4 \end{bmatrix} = \begin{bmatrix} 1/5 & 1/15 \\ -1/10 & 2/15 \end{bmatrix}$

14. Find all a, b for which the following system has no solutions.

$$\begin{aligned} -x + 3y &= 5 \quad \times(-4) \\ 4x + by &= a. \end{aligned}$$

$4x - 12y = -20$

As long as $b = -12$, there will be ~~no~~ solutions if $a \neq -20$

15. Suppose $M = \begin{bmatrix} 1 & -1 & 3 \\ 2 & 0 & 4 \\ 7 & 1 & 6 \end{bmatrix}$. Find M^T, the transpose of M.

$$\begin{bmatrix} 1 & 2 & 7 \\ -1 & 0 & 1 \\ 3 & 4 & 6 \end{bmatrix}$$

16. Find the determinant of A, if

$$\begin{bmatrix} 1 & 1/3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & -1/3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} A = I,$$

$\uparrow$ This is D^{-1} for LDU

So, $D = \begin{bmatrix} 1/4 & 0 \\ 0 & -3 \end{bmatrix}$ and $\det A = \det D = -3/4$

17. You are finding the LDU decomposition of a 3×3 matrix A. After the first two steps, you know so far that the matrix $e_{31}^{3} e_{21}^{-1/2} A$ results in the system

$$\begin{aligned} 4x - y + 3z &= 8 \\ -3y + 2z &= 10 \\ 12y - z &= 4. \end{aligned} \quad \times 4$$

What are the next two matrices you will need to compute? In other words, as you continue to $S_2 S_1 e_{31}^{3} e_{21}^{-1/2} A$, what are S_1 and S_2?

$$\begin{aligned} 4x - y + 3z &= 8 \\ -3y + 2z &= 10 \\ 7z &= 44 \end{aligned}$$

1^{ST}, $S_1 = e_{32}^{4} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 4 & 1 \end{bmatrix}$

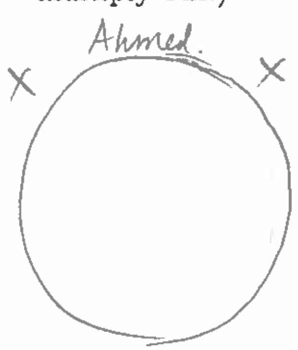
Then divide by the coefficients: $S_2 = D = \begin{bmatrix} 1/4 & 0 & 0 \\ 0 & -1/3 & 0 \\ 0 & 0 & 1/7 \end{bmatrix}$

18. How many distinct words can be made from the letters in PINEAPPLE?

9 letters, 3 P's, 2 A's: $\binom{9}{3\ 2\ 1\ 1\ 1\ 1} = \dfrac{9!}{3!2!}$

$= 30,240$

19. Ten people (include Ahmed and Beatrice) sit around a circular table. How many ways can they be seated if Ahmed and Beatrice do *not* sit next to each other? (No need to multiply out.)



Place Ahmed. That leaves 7 spots for Beatrice. The remaining 8 can be in any order.

$7 \cdot 8!$

20. How many ways are there to get a flush in Poker? Assume there is no Joker.

$$\underset{\text{Choose suit}}{\underline{4}} \times \underset{\text{Choose 5 cards}}{\underline{\binom{13}{5}}} - \underset{\text{Choose suit}}{\underline{4}} \times \underset{\text{Choose beginning card}}{\underline{10}}$$

You must subtract the straight flushes.

21. What is the probability that a five-card Poker hand contains the King of Hearts?

To have the K♡, you must have 4 other cards from the remaining 51.

$$\frac{\binom{51}{4}}{\binom{52}{5}}$$

22. An 8-sided die (with the numbers 1–8 on the faces) is loaded in such a way that the probability of each face turning up is proportional to the number of dots on that face. (For example, a six is three times as probable as a two.) What is the probability of getting an even number in one throw?

Probabilities:	1	2	3	4	5	6	7	8
	1/36	2/36	3/36	4/36	5/36	6/36	7/36	8/36

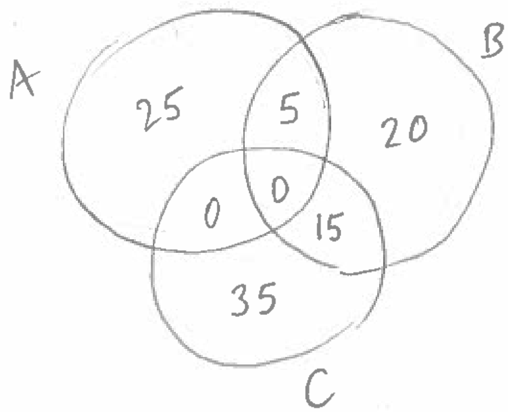
On one throw: Even $= \frac{2+4+6+8}{36} = \frac{5}{9}$

23. If $P(A \cap B) = 1/13$, $P(A \cup B) = 9/13$, and $P(\widetilde{A}) = 10/13$, what is $P(B)$?

$P(A) = 1 - P(\widetilde{A}) = 3/13$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$
$$9/13 = 3/13 + P(B) - 1/13$$
$$P(B) = 7/13$$

24. Several students are taking Art, Biology, and Chemistry. 5 students take Art and Biology, 15 take Biology and Chemistry, but no one takes both Art and Chemistry. 30 students take art. The number of students taking Chemistry is 10 more than the number taking Biology, and the number taking Biology is 10 more than the number taking art. How many students are there altogether?



There are $25+5+20+15+35$
$= 100$ students

25. One one six-sided die, there are 4 faces with odd numbers on them, and 2 with even numbers. On a second six-sided die, there are 3 faces with odd numbers on them, and 3 with even numbers. Each face is equally likely to come up on each die. You roll the two dice, and add the faces. *Using generating functions,* count how many possible rolls there are with an odd sum.

use x^0 for even, x^1 for odd. (parity)

$$(2x^0+4x^1)(3x^0+3x^1)$$

$$6x^0+18x^1+12x^2$$

$\uparrow$

18 possible rolls with an odd sum

26. B is uniformly distributed on $[0,1]$. Find the probability that $|B-1/3| \le 1/6$.

$$-1/6 \le B - 1/3 \le 1/6$$

$$\frac{1}{6} \le B \le \frac{1}{2}$$

$$\frac{1}{2} - \frac{1}{6} = \frac{1}{3}$$

27. A square board is divided into equal quadrants, colored black and white as shown. The board is 20 cm wide. You toss a coin of radius 1 cm onto the board. What is the probability that it lies entirely on the board and *only* on one color? In other words, the coin cannot go past the edges or cross a line between black and white quadrants.

Leave a 1 cm border in each quadrant, So there is an 18 cm square for the center of the coin to land in.

$$\frac{8^2}{9^2} = \frac{64}{81}$$

28. You are rolling an 8-sided die (each face 1–8 is equally likely to come up). A is the event that you roll a 1, 2, or 3. B is the event that you roll an even number.

    (a) Find $P(A|B)$.

    (b) Find $P(B|A)$.

    (c) Are A and B independent? Justify your answer.

$P(A) = \frac{3}{8}$ $P(B) = \frac{1}{2}$ $P(A \cap B) = \frac{1}{8}$

a) $P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{1/8}{1/2} = \frac{1}{4}$

b) $P(B|A) = \frac{P(B \cap A)}{P(A)} = \frac{1/8}{3/8} = \frac{1}{3}$

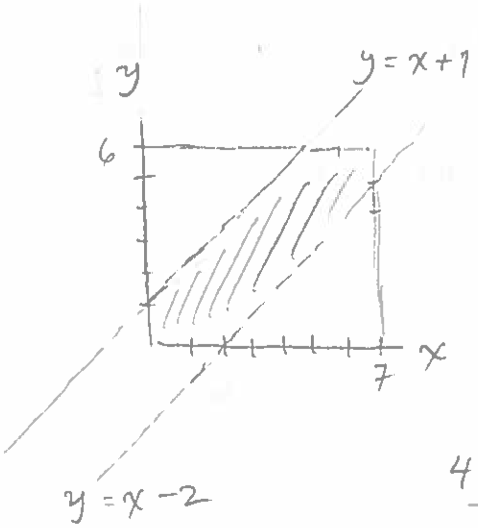
c) $P(A)P(B) = \frac{3}{8} \cdot \frac{1}{2} = \frac{3}{16} \neq P(A \cap B)$

A and B are <u>not</u> independent.

29. You and a friend were both at the park, but you didn't know it! You were there for one hour, and your friend was there for two hours. You both were there sometime between 1:00 and 9:00 (arriving after 1:00 and leaving before 9:00). Assuming uniform arrival times, what is the probability that you were there at some time that your friend was also there?

Let x = your arrival time
y = your friend's arrival time

Let 0 be 1:00
8 be 9:00



You arrive before your friend leaves:

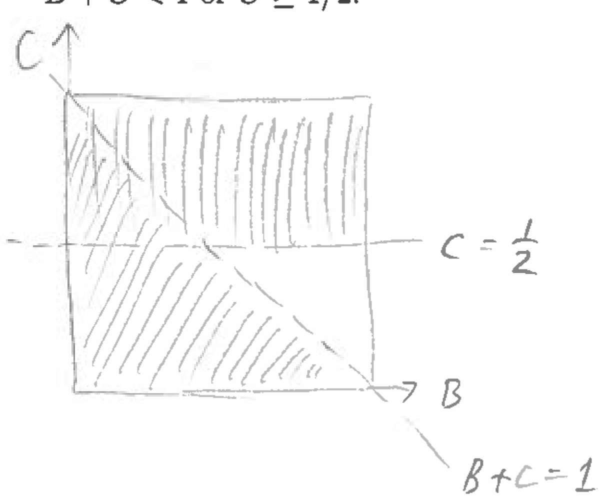
$x \leq y + 2$

Your friend arrives before you leave

$y \leq x + 1$

$$\frac{42 - 2(\frac{1}{2})5 \cdot 5}{42} = \frac{17}{42}$$

30. B and C are uniformly distributed on the interval $[0, 1]$. Find the probability that $B + C < 1$ or $C \geq 1/2$.



OR: Union of the two areas

$$\frac{7}{8}$$

EXTRA CREDIT:

1. You have two 4-sided die. One has faces numbered 1–4, and the other has faces numbered 5–8, with each face being equally likely to come up. It is possible to find two 4-sided dice with *different* numbers (positive integers) on the faces so that the probabilities of getting the rolls 6–12 are *exactly* the same as with the original 4-sided dice. Find the unique way to do this!

2. A large group of friends are going to dinner, and they will sit at a very big round table. Ten of them are wearing blue hats, six are wearing green hats, and five are wearing red hats. How many different ways can they sit so no one sits next to someone with the same color hat? For example, no two people with blue hats can sit next to each other.

3. Suppose M is a 2×2 matrix such that $M \neq \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ and $M^2 = 0$.

   (a) Show that 0 is an eigenvalue of M.

   (b) Show that there are no other eigenvalues of M.

   (c) What is the algebraic multiplicity of 0?

   (d) What is the geometric multiplicity of 0?