On Problem Writing

No educator disagrees that problem solving deserves attention in any curriculum. (See Standard M 6.1.) Of course there is good reason for this, as much of what we do in our daily lives involves problem solving to some degree.

There is no consensus on how to teach students how to solve problems, although much has been written on the subject. One of the earlier sources is the classic *How To Solve It,*[1] where Polya lists the basic steps as (1) Understanding the problem, (2) Devising a plan, (3) Carrying out the plan, and (4) Looking back. Polya then goes on to discuss these ideas further, as well as provide a glossary of important terms and concepts in problem solving in mathematics (many of which are applicable to problems in other areas as well.)

The challenge, pedagogically, is to foster the development of mathematical *intuition.* Consider Step (2), "Devising a plan." How is a student to do this? One of my colleagues, Michael Keyton, makes the important distinction here between a "problem" and an "exercise."

Let us suppose you teach the students the quadratic formula; that is, to solve the equation $ax^2 + bx + c = 0$, where $a \neq 0$, you calculate

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

to arrive at the solutions. You then do an example, after which you give your class a few quadratic equations to solve.

There is no need to devise a plan here – the calculation is routine. This is an exercise, not a problem. The student recognizes a quadratic equation, knows that there is a technique for solving such an equation, and then applies the technique.

Once students learn the quadratic equation, they are able to solve more complex problems, such as:

- Solve $x^6 - x^3 - 1 = 0$ for x.
- What numbers are one greater than their reciprocals?
- Given a regular pentagon, find the ratio of the length of the diagonals to the length of the sides.

[1] G. Polya, ISBN 0-691-02345-5, 1945.

These problems, which require using the quadratic equation for their solution, are not merely applications of a formula to a given equation (although if you just showed students how to solve $x^4 - 3x^2 + 2 = 0$, the first example would be an exercise).

Would students be able to solve these problems? How? Some degree of intuition is required here. Can a typical student see the first problem as the disguised quadratic

$$(x^3)^2 - (x^3) - 1 = 0$$

without having seen a similar example first?

Doing a few examples and then having students work out similar examples is not teaching students problem solving, but rather showing students techniques which they may then use to work exercises. Of course this is important in any classroom. Although it might be possible to create interesting series of problems which would enable students to learn *any* important idea, this method of teaching takes more class time than traditional instruction.

I tell my students that problem-solving intuition can be developed by simply doing *lots* of problems. Naturally, these need to be genuine *problems* for the students – not simply exercises, but also not problems which are impossible for students to solve at their level of mathematical ability. This requires a high degree of motivation on the part of the student; many of us who teach mathematics have experienced the delight in routinely tackling difficult problems successfully.

Thus it is impractical simply to give students a large bank of problems and expect them to solve them on their own. And given curricular constraints, few teachers have the luxury of spending significant class time devoted to working problems. (Although I am fortunate enough to be able to teach a course called *Advanced Problem Solving* here at IMSA, where much of our class time does in fact focus on problem solving.)

One approach that I have found particularly effective in teaching problem-solving is to have students *write* problems. This can be done in a variety of ways.

The most open-ended way is simply to give the students the assignment to "write a problem." I take this approach in *Advanced Problem Solving.* Of course students may write problems based on current work in the course, but many students take this opportunity to explore other areas of mathematical interest.

I have found this approach successful as my background includes several years of teaching at university. I am familiar with most topics students find suitable for problem writing, although I have been stumped on occasion and have needed to ask a student for further explanation of a problem.

But giving open-ended problems does not necessarily depend on background. Suppose that a student writes a problem involving number theory, while the teacher has not studied the subject in depth. A confident teacher can critique the writing style and the presentation of the problem, and then say, "I am not sure, but let's explore this together." This requires a teacher's ability to say "I don't know the answer," as well as the time to work with the student more informally.

Although more challenging, there is an important benefit to taking this approach. To illustrate, I excerpt a few sentences from a reflection I ask my *Advanced Problem Solving* students to write on their progress as a problem writer over the semester:

> Anyone can write tedious, difficult problems that review core math subjects, but to write problems in a novel, challenging, and refreshing manner, one must be imaginative. I feel that this creative side of math is an often overlooked aspect of the field as many believe math to be an extremely black-and-white, rigid, and boring subject.

Reading this for the first time was quite a surprise for me. Although I had not specifically intended to teach students how to appreciate creativity in mathematics, I found that this was a pleasant side-effect of having students write original problems.

There are less open-ended ways to have students write problems. A second approach is one I occasionally use in our precalculus courses. I ask students to write a quiz or an exam on a particular subject. They must also write a brief reflection about the choice of problems they included on their quiz. This task is more focused, and consists mainly of writing exercises rather than problems. But it is a good introduction to problem writing for students, and some students really write some very creative problems. Moreover, since the topic is one the students are currently learning (or have recently learned), the teacher has the background necessary to give meaningful feedback on student work. If the assignment is a longer one (such as writing an entire exam), I let students work in pairs, which gives them an opportunity to work collaboratively. I must add that as a result of completing this assignment, students often remark that they have a greater appreciation of teachers and the work it takes to create a good exam.

A third approach is to give a more focused assignment, such as "Write a problem which requires using the Pythagorean Theorem to solve it." Narrowing the focus this far is often a help to students unfamiliar with writing problems, and is also good for teachers new to using problem-writing in the classroom.

Of course there are many variations on these themes. As a teacher becomes more experienced in using problem writing in the classroom, he or she may come up with different projects for students to work on.

Now does having students write problems actually help them improve their problem-solving abilities? My students tell me that it does – although admittedly the sample size is small. I can also state that the conversations I have with students about writing problems are of a different character than those about solving problems – they are of a more abstract nature. They are about the *structure* of problems, rather than about *techniques* of problem solving.

So let us assume that having students write problems is in fact beneficial – as I am convinced it is. Now that some ideas for projects have been mentioned, an important question arises: how should problem writing be discussed in the classroom? I can say from experience that simply telling students to write problems without any guidance is often not very effective....

There are several ways to bring problem writing into the classroom. One simple way is to have students bring in a few favorite problems. They can show their problems to the class, and be asked why they like them. This naturally leads into a discussion of the question, "What makes a problem interesting?" Of course there is no single answer to this questions, as different students will undoubtedly like different types of problems. But this might be the first time students have *discussed* problems rather than *solve* them.

Another approach would be for the teacher to find an example of a problem which could lead into a discussion of problem design. Consider the following straightforward geometry problem:

> One side of a rectangle is 8 cm long, and the diagonals are 17 cm. Find the area of the rectangle.

Students should easily be able to get 120 cm^2. Now they might be asked, "Why was the solution an integer?" This leads to a discussion of Pythagorean triples, which moves the discussion to number theory. So how do you create Pythagorean triples? A brief Internet search (such as in the English Wikipedia) reveals that any Pythagorean triple (a, b, c) may be generated by positive numbers m and n, with $m > n$, according to the formula

$$a = m^2 - n^2, \quad b = 2mn, \quad c = m^2 + n^2.$$

The proof of this formula requires familiarity with number theory, and so is beyond the ability of most students. But they can verify that $a^2 + b^2 = c^2$ once m and n are given as specified, and they could certainly change the numbers in the geometry problem above so that the answer is an integer.

Once this is done, it is possible to further alter the problem as follows:

> The lengths of the sides and the diagonals of a rectangle are all integers. The perimeter is 46 cm. What is its area?

This is a much more interesting and challenging problem. Moreover, in solving such a problem, the student should be able to show that their answer is unique.

Now showing uniqueness might be done by simply doing all the necessary calculations. But consider the possibility that the sides have lengths 6 and 17 cm. It is easy to see that $\sqrt{6^2 + 17^2}$ is not an integer. But an interesting discussion might follow by considering the formula for generating Pythagorean triples.

In the formula, a may be even or odd depending upon the values of m and n. But b is necessarily even, so we must have

$$2mn = 6.$$

Since we assume that $m > n$, the only possibility is $m = 3$ and $n = 1$. Substituting into the expressions for a and c results in $a = 8$ and $c = 10$. Thus, we have proved that if 6 is a leg of a right triangle with integer sides, the other leg must be 8 and the hypotenuse must be 10. Thus, it is impossible for the diagonal of a rectangle to be an integer if the sides are 6 and 17.

Now this is a long digression on a simple geometry problem. But it illustrates the point that a carefully chosen problem can be the starting point for a discussion on problem writing. And as it is fairly common that the numbers in a given problem are chosen so that the problem "works out" nicely, it should not be too difficult to find suitable examples.

Another possible idea is to use students' own problems as examples. I have had success with discussing student work in class, with the name of the student omitted if he or she wishes. I have found that students are respectful of each other's work, and make appropriate comments. If the teacher can create the right classroom environment, this way of discussing problem writing can be very effective.

In taking this approach, you will want to select a variety of problems. Sometimes a student creates a problem that others think is too simple. Rather than view this as a detriment, it is possible to bring up the question of the *audience* for a particular problem. If you are in an eighth-grade geometry class and a student writes a simple problem, ask the class what grade level they think the problem is appropriate for. This can generate an interesting discussion. It is an important point, for I have often heard the comment from beginning problem writers that they tend to create problems that are either too easy or so difficult that they cannot solve them.

Perhaps the most challenging strategy for a teacher is a brainstorming session. Ask students to think out loud about problem ideas. I have found it useful to ask a student to come to the board and draw some geometrical figure. Then ask, "How can we create a problem out of this?" The challenge is that when students come up with ideas, the teacher should be able to guide them along a fruitful path; it is helpful to be able to have a sense that one idea may not really lead anywhere, while another might move the class into interesting territory. Guiding such a classroom discussion takes some experience.

One suggestion for teachers may be to come up with a figure in advance – say a right triangle with certain side lengths inscribed in a circle. Then the teacher can spend some time creating problems relating to this figure. When it comes to the class discussion, the teacher will have some idea of what students will suggest, and so will be able to offer useful guidance.

These are the classroom strategies I have used in the past, and they have usually worked out well. Of course there are many others – whatever the choice, it is important that the teacher feel confident in their ability to direct the classroom discussion. As teachers work more and more with problem writing, their level of confidence will certainly increase.

What about the assignments themselves? What prompts are given to students to write problems? I consider writing an original problem to be a *writing* assignment, so that they must use complete sentences in paragraph form for all of their work.

For my students in grades 10–12, I usually ask them to include the following:

- Motivation: Just a sentence or two telling me what gave them the idea for their problem. Was it one they found in a book? Or a doodle they were making?
- Problem Statement: This should be clear and unambiguous. Students sometimes make assumptions which they forget to explicitly state in their problem, and then the problem is ill-posed or too difficult.
- Problem Solution: This should be written in paragraph form. I do encourage students to submit partial solutions if they make a very difficult problem. It is a good learning experience for them. They may also revise the problem and submit it again for a subsequent assignment.
- Reflection: This only needs to be a few sentences. This is a chance for students to say something about what they learned about writing or solving problems as a result of having done the assignment. In *Advanced Problem Solving,* I have students put together a portfolio at the end of the semester, and so having these frequent reflections is helpful.

I propose conducting some workshops based on these ideas. All of the classroom ideas mentioned above – discussing favorite problems, brining in model problems, critiquing problems, and brainstorming – can be modeled in a workshop setting. In addition, such a workshop would be an excellent way to illustrate teaching at IMSA, as I would be using many teaching strategies that I currently use in my classes.

The accompanying document on pentominoes is intended to supplement these ideas. I would suggest this topic for such workshops as it lends itself to working with teachers of all grade levels. The topic is accessible enough so that teachers at lower grade levels can understand the basic ideas, but rich enough that very challenging problems can be created for even the brightest students.