Original Problem

Problem: An increasing sequence $1, 99, 100, 9801, 9802, \ldots$ consists of all positive integers that are sums of distinct powers of 99. If one of the first 99^{100} terms is chosen at random, what is the probability that it is a multiple 100?

Motivation: The idea for this problem cam from one that I solved as part of a qualifying quiz: "The increasing sequence $1, 3, 4, 9, 10, 12, 13, \ldots$ consists of all positive integers that are sums of distinct powers of 3. Of the first googol (10^{100}) terms in the sequence, how many are powers of 3?" Although that problem can be solved in multiple ways, one way of thinking about it was extremely elegant and I wanted to use that thought process in a different way.

Solution: The numbers in the sequence are all integers that can be represented with only 1s and 0s in base 99, and 100 in base 99 is 11. Since only 1s and 0s are used, after we convert everything into base 99, for our purposes we can treat the numbers as if they are in base 2. Thus, the problem is equivalent to finding the probability that a number that can be represented as the sum of distinct powers of 2 chosen at random from the first 99^{100} such numbers is a multiple of 3.

Any positive integer can be written as the sum of distinct powers of 2, so this is the same as the probability that an integer chosen at random from the first 99^{100} positive integers is a multiple of 3, which is $\frac{1}{3}$.

Furthermore, every third integer is divisible by three, so every third integer in the original sequence will be divisible by 100.

My Best Problem #2

Consider the set of all four-dimensional unit hypercubes with one vertex at the origin. Calculate the hypervolume of the envelope of this set of hypercubes.

Solution

We note that the main diagonal of a unit hypercube has length $\sqrt{1^2+1^2+1^2+1^2} = 2$. The envelope formed therefore consists of the set of points at distance 2 from the origin. The envelope is therefore a 4-sphere of radius 2.

We first calculate the hypervolume area of the 4-sphere by considering a slicing of the 4-sphere into 3-spheres. At a distance r from the center of the 4-sphere, an intersection of a hyperplane with the 4-sphere yields a 3-sphere of radius $\sqrt{R^2-r^2}$. The differential volume of such a 3-sphere can be written as $\frac{4\pi}{3}(R^2-r^2)^{\frac{3}{2}}\,dr$. In order to calculate the total hypervolume of the 4-sphere, we integrate the differential volume from 0 to R and subsequently double the result.

We integrate to determine that

$$\int \frac{4\pi}{3}(R^2-r^2)^{\frac{3}{2}}\,dr = \frac{\pi}{3}\left(\frac{3R^4\sin^{-1}\left(\frac{r}{R}\right)}{2} + \frac{r(R^2-r^2)^{\frac{3}{2}}}{4} + \frac{3R^2r\sqrt{R^2-r^2}}{2}\right)$$

Introducing our limits of integration, 0 to R, we obtain

$$\int_0^R \frac{4\pi}{3}(R^2-r^2)^{\frac{3}{2}}\,dr = \frac{\pi}{3}\left(\frac{3R^4\sin^{-1}(1)}{2}\right) = \frac{\pi R^4}{2}\cdot\frac{\pi}{2} = \frac{\pi^2R^4}{4}$$

Doubling to cover the entire hypersphere, we obtain a hypervolume of $\frac{\pi^2R^4}{2}$. Finally, substituting $R=2$ into this expression yields a net hypervolume of $\mathbf{8\pi^2}$.

Strong Problem 2: What bases can 4174 in base ten be converted to so that the digits of the new base form a three letter word?

Solution: We recognize a three digit base expansion has the form

$$d_1 b^2 + d_2 b + d_3$$

where d_1, d_2, d_3 are the three digits and b is the base. Since the letters of the alphabet correspond to the numbers 10 to 35,

$$10b^2 < 4174$$

in order for the first digit to be a letter. Thus, we can see that $b \leq 20$. We also need to ensure that the base is not so small so that the conversion of 4174 is greater than 3 digits, or

$$b^3 > 4174.$$

Thus, $b \geq 17$. Since $17 \leq b \leq 20$, we now only need to test 4 cases. After testing, we find that $b = 19$ and that 4174 in base nineteen is BAD.

Motivation: I was interested by the discussion on alternate bases in class and I wanted to gain familiarity with bases over ten. Also, I wanted to incorporate some of the wordplay found in the book problems.

APS Original Problem

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Problem

Solve

$$\begin{array}{r} NINE \\ +ONE \\ \hline ZERO \end{array}$$

Given that each letter represents a different number and $O = 6$

Solution

We are given $O = 6$ and we know that $E + E$ has a units digit of 6, so $E = 3$ or 8 . Let us assume that $E = 8$. Looking at the column containing the hundreds digits, we see that $I + O = E$, so $I + 6 = 8$. So, the only possible value for I is 2. or 1, if there was a carry! However, looking at the next column, we see that $N = Z$. This means that there must have been a carry in the hundreds place, which is a contradiction. Therefore, we know that our original assumption is wrong, and that $E = 3$.

We see that $I + O = E$, or $I + 6 = 3$. We know that it must be $I + 6 = 13$ because that is the only possible value for $I + 6$ that has a units digit of 3. Therefore, we know that $I = 6$ or 7. However, I cannot equal 6 because $O = 6$. So, $I = 7$.

Now, we know that $N + N < 10$ because it could not have carried 1 into the hundreds place. So, we know that $N = 1,2,$ or 4. Let us assume that $N = 1$. That would make $R = 2$ and $Z = 2$. This cannot happen so we know that $N \neq 1$. Now, let us assume that $N = 2$. This would make $R = 4$ and $Z = 3$. However, we already know that $E = 3$, causing a contradiction. Therefore, we know that $N = 4$. This would make $R = 8$ and $Z = 5$.

10/10

Thus, we have solved our cryptogrythm.

$$\begin{array}{r} 4743 \\ +643 \\ \hline 5386 \end{array}$$

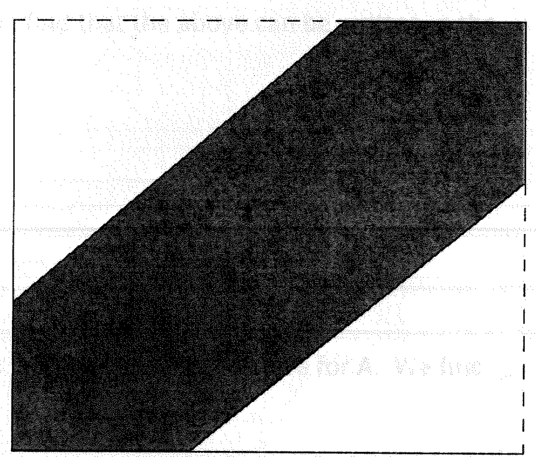
good work

Problem:

Every day between 5:00 PM and 6:00 PM, two trains go between Hinsdale and Chicago on the same track, one going to Chicago and one coming from Hinsdale. The trip takes exactly 20 minutes. The first train leaves Chicago at 5:00 and the second train leaves Hinsdale at 5:30 so that the trains won't collide. A mischievous IMSA student decides to change the conductor's watch on both trains so that each train will leave at a random time between 5:00 PM and 6:00 PM. What is the probability that the trains will crash between 5:00 PM and 6:00 PM?

Solution

We can solve this problem by using geometric probabilities. Let us create a graph, denoting the x- axis as the time elapsed after 5:00 PM that the first train leaves Chicago. The y-Axis will represent the times that the second train could leave so that the two trains would crash. We find that the two trains will crash if the second train leaves between:
$y = x + 20$ and $y = x - 20$. The area of the shaded region, that is the region where the two trains will crash, is: $(60)(60) - 2\left(\frac{1}{2}\right)(40^2) = 2000$



Since the total area is $(60)(60) = 3600$, the probability of a crash is:

$$\frac{2000}{3600} = {}^{5}/_{9}$$

+25
to Problem Set
3

Problem

There are n evenly spaced points that lie on a circle of radius 2009. If two segments are chosen, each of them connecting exactly two of these points and neither of them going through the same point, what is the probability that the two segments will intersect?

Solution

The second part of the question tells us that we must choose four distinct points in order to have two segments. For each group of four points, they can be connected in three different ways. For example, given points A,B,C,D, point A could be connected to either point B,C, or D while the two remaining points form the other segment. In only one of these three cases do the segments intersect. Therefore, the probability that the two segments intersect is $\frac{1}{3}$. The radius of the circle in this problem is completely irrelevant and put there just to trick the reader.

Motivation

This problem came to me after seeing the friend wheels on Facebook.com, in which one's friends are put on a circle and mutual friends are connected via different colored line segments. After seeing countless intersecting segments, I decided to try to find out the probability that two segments would intersect.

OK – but take this idea a bit further – create a more challenging problem!

9/10

Problem

Given the two recursively-defined sequences, prove that there are no common terms between the two sequences.

$$G_{n+2} = 2G_{n+1} + 3G_n$$

and

$$H_{n+2} = H_{n+1} + 5H_n$$

The initial terms for the sequences are $G_0 = 1$, $G_1 = 1$ and $H_0 = 0$, $H_1 = 8$.

Solution

If we take both sequences modulo 8, we see that the first sequence repeats 1,1,5,5,1,1,5,5,... while the second sequence is always congruent to 0 (mod 8), as both initial terms are divisible by 8. Therefore, the two sequences share no common terms.

Nice!

10/10

Motivation

After seeing the first student-written problem presented in class, I wanted to come up with something similar: a problem that was concise and short, but required a tricky insight. The best thing I could think of was using modular arithmetic and combining it with recursive sequences, something we did recently.

48/50

Best Problem 1

A paper written by an alien from the planet Spdzk was discovered by scientists on Earth. From this paper, it appears that several things have different colors on Spdzk than they do on Earth. In this paper, two sources are cited and two statements are uncited. According to the teacher's comments, all of the statements that from one source were lies, while all of the statements from one source were true. Furthermore, one of the uncited statements was true, while the other was false. The teacher added that frogs are either blue or green and that grass is either blue or green. Based on the following statements, assuming that the teacher was correct, what colors are frogs, grass, and the sky on Spdzk?

Statements cited from the Spdzk Book Encyclopedia:
If Spdzkipedia is telling the truth, then the sky is blue.
The sky is not blue and grass is not green.

Statements cited from Spdzkipedia:
Frogs are not blue if and only if the sky is blue.
Grass is green or frogs are green.

Uncited statements:
Frogs are the same color as the sky.
If frogs are a different color from the sky, the sky is purple.

Best problem 2

Alice, Betty, and Carol are playing a game with six rounds. Two of the girls play each round. At the end of a round, the girl who was not playing replaces one of the girls who was not playing. Alice and Betty both play during round 1. What is the probability that they are both playing during round 6?

Worst Problem

Original Version

A group of Earthlings met up with a Spdzkian band. The Spdzkians said that they were waiting for their gdrp player. The Earthlings were intrigued and wanted to know more about this instrument. Each of the four members of the band said something about the gdrp, but their statements were rather

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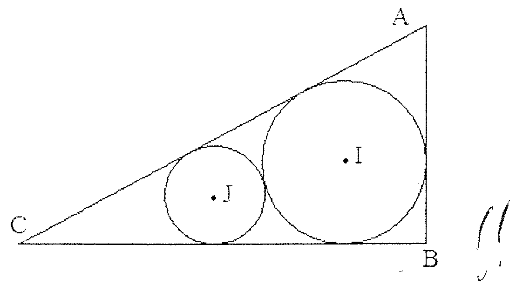
Problem Set 7

Original Problem 10

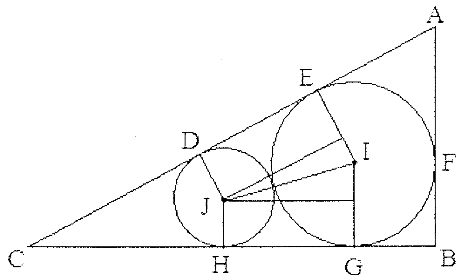
Problem



In the diagram above is right triangle ABC with two inscribed circles on it's interior. Given that the radius of circle I is 4, the radius of circle J is 1, and side $\overline{BC} = 20$, find the area of the triangle. Express your answer in reduced improper fraction form.

Motivation
I think that walk-around problems are really darn cool, so I decided to take a single circle walkaround problem to the next level by inscribing two circles inside of a triangle instead of only one. This type of problem makes for an interesting non-trivial solution.

Solution



OK. You're figure is way not to scale ? Why?

Begin by drawing in the lines shown in the image above. First of all, we know that because of 3-4-5 right triangles, that segments $\overline{DE}$ and $\overline{HG}$ have a length of 4. This is because the sum of the radii of the circles is 5, and the difference is 3. Therfore, when we draw segments parallel to sides $\overline{AC}$ and $\overline{BC}$, forming a right angle with these sides, we form the said 3-4-5 triangles. From here, we know that $\overline{GB}$ and $\overline{FB}$ are also both equal to 4 because when lines $\overline{GI}$ and $\overline{FI}$ form a square with sides $\overline{GB}$ and $\overline{FB}$ due to the right angle in the triangle and law of 2 tangents. Now, because we know that $\overline{HB}$ has length 8, and since $\overline{BC} = 20$, $\overline{CH} = 12$. By the law of two tangents, and our previous statement, $\overline{CD}$ also equals 12. Now, let us assign a variable to the sides that we do not know yet. We still do not know what the length of $\overline{FA}$ is, so let us call this value x. Now, by the law of two tangents, again, $\overline{EA} = x$, because $\overline{FA} = x$. Now, in order to find the area of the triangle, we simply need to solve for x, find the length of side $\overline{AB}$, and substitute all of these values into the formula for area. We can use the pythagorean theorem to solve for x, because ABC is a right triangle.

$$\begin{aligned}
(4+x)^2 + 20^2 &= (16+x)^2 \\
16 + 8x + x^2 + 400 &= 256 + 32x + x^2 \\
160 &= 24x \\
x &= \frac{40}{3}
\end{aligned}$$

good problem!
10/10

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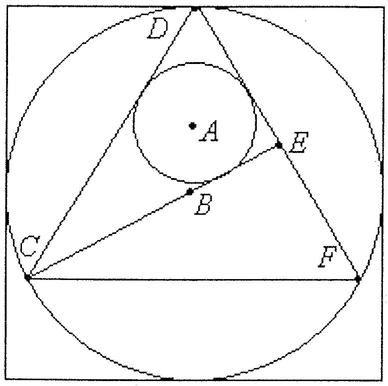
Problem Set 7

Original Problem 4

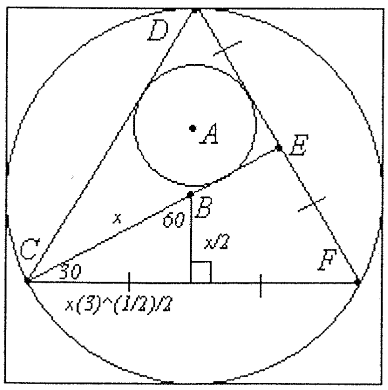
Problem



equilateral In the diagram above is a square, circumscribed about circle B. Circle B then circumscribes CDF, which is a triangle that is bisected by line segment CE. Circle A is inscribed in CDE. If the side length of the square is 2x, determine what the radius of circle A is.

Motivation
I decided that I wanted to make a geometry problem that I could actually solve before writing. Although it doesn't quite measure up in difficulty level to the ones that I already wrote, but it is still an interesting problem. It is a good warm-up or practice problem for a geometry related contest.

Solution
We know that since circle B is inscribed in the square, half of the side length of the square must be equal to the radius of circle B. We know that the radius is x. Continuing on, we now have to determine what the side lengths of the triangle is in order to determine the side lengths of the triangle circumscribing circle A. If we draw a line down from point B, perpendicular to the base of CDF to a point P, we form a 30-60-90 triangle relationship.



From here, we know that the side length of the triangle must be $\frac{x\sqrt{3}}{2}(2) = x\sqrt{3}$. In order to find the the inradius of circle A, we need to know what the perimeter, P, of triangle CDE is, and it's area. These can be easily found by using the 30-60-90 relation. We know that DE is equal to $\frac{x\sqrt{3}}{2}$, side CD is twice that,