# Generic Ellipses as Envelopes

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A curve $C$ is said to be an *envelope* of a family of curves if each curve of the family is tangent to $C$. The astroid, described by the equation $x^{2/3} + y^{2/3} = k^{2/3}$ $(k > 0)$, is well known [**1**] as being generated by two different families of curves. The first is a family of line segments, as in Figure 1. A line segment of length $k$, sliding without slipping so that its endpoints lie on the coordinate axes (imagine a ladder sliding down a wall), remains tangent to an astroid. The astroid is also the envelope of a family of concentric ellipses (as in Figure 2), where the sum of the lengths of the axes is the constant $2k$.

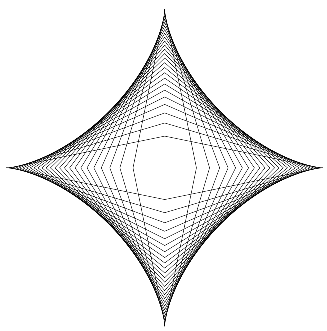


Figure 1

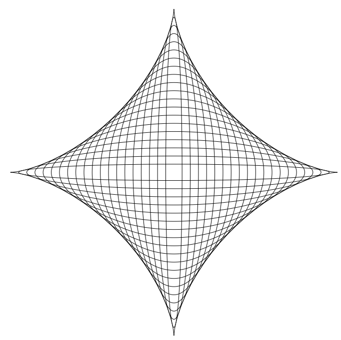


Figure 2

As it happens, there is a surprising connection between these two figures, which will be revealed in the main result of this paper. A related diagram is one produced by a typical grade school art student on a piece of graph paper: in the first quadrant, draw lines such that the sum of the $x$- and $y$-intercepts is some constant $k > 0$, then reflect to the other quadrants. This results in Figure 3, which can be described by $|x|^{1/2} + |y|^{1/2} = k^{1/2}$. Elimination of roots from this equation when $x, y \geq 0$ results in

$$x^2 - 2xy + y^2 - 2kx - 2ky + k^2 = 0,$$

a parabola. Only the part of this parabola with both $x \leq k$ and $y \leq k$ is seen in the first quadrant of Figure 3.

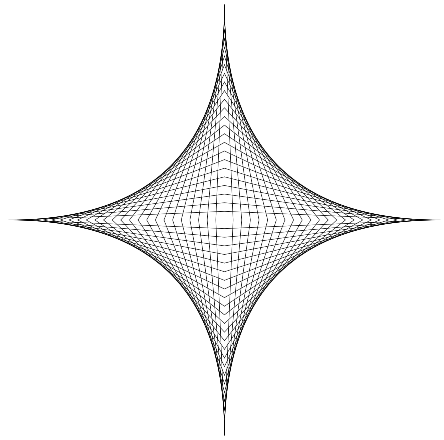


Figure 3

What do these figures have in common? For $a, b, m > 0$, define a *generic ellipse* $E(a, b; m)$ to be the set of points satisfying

$$\left|\frac{x}{a}\right|^m + \left|\frac{y}{b}\right|^m = 1.$$

We refer to the four points $(\pm a, 0)$ and $(0, \pm b)$ as the *vertices* of the generic ellipse. Note that tangents exist at the vertices only when $m > 1$. When $m = 1$, the vertices are the four corners of a diamond, and are cusps when $0 < m < 1$. We refer to $E$ as an *$m$–ellipse*.

Each of the figures above illustrates a result about the envelope of a family of generic ellipses. For example, Figure 2 is a graphical solution to the question: What is the envelope of the family of 2–ellipses $E(a, k - a; 2)$, where $k > 0$ and $0 < a < k$?

In seeking an equation for such an envelope, a natural question to be asked is: When are two generic ellipses $E_1(a_1, b_1; m_1)$ and $E_2(a_2, b_2; m_2)$ tangent? First note that tangency at vertices may only occur if both $m_1, m_2 > 1$ and either $a_1 = a_2$ or $b_1 = b_2$. When $m_1 = m_2 > 1$, $E_1$ and $E_2$ may *only* be tangent at vertices (unless they are the same $m_1$–ellipse), as we will see later.

In turning our attention to tangency at points other than vertices, we assume that $m_1 \neq m_2$. Due to symmetry, we consider the case when $E_1$ and $E_2$ are tangent at $(x_0, y_0)$ in the first quadrant, so that $x_0, y_0 > 0$.

A routine calculation reveals that the tangent lines to $E_1$ and $E_2$ at $(x_0, y_0)$ are given by

$$\frac{xx_0^{m_i-1}}{a_i^{m_i}} + \frac{yy_0^{m_i-1}}{b_i^{m_i}} = 1, \quad i = 1, 2.$$

$E_1$ is tangent to $E_2$ at $(x_0, y_0)$ if these lines are the same, so that

$$\frac{x_0^{m_1-1}}{a_1^{m_1}} = \frac{x_0^{m_2-1}}{a_2^{m_2}}, \quad \frac{y_0^{m_1-1}}{b_1^{m_1}} = \frac{y_0^{m_2-1}}{b_2^{m_2}}. \tag{1}$$

(Note that if $m_1 = m_2$, and if $E_1$ and $E_2$ were tangent at a point which is not a vertex, so that $x_0 \neq 0$ and $y_0 \neq 0$, then (1) would imply $a_1 = a_2$ and $b_1 = b_2$, and hence $E_1$ and $E_2$ would be identical.)

Now we may solve for $x_0$ and $y_0$ to determine the purported point of tangency:

$$(x_0, y_0) = \left( \left( \frac{a_1^{m_1}}{a_2^{m_2}} \right)^{\frac{1}{m_1-m_2}}, \left( \frac{b_1^{m_1}}{b_2^{m_2}} \right)^{\frac{1}{m_1-m_2}} \right). \tag{2}$$

Of course it is necessary that $(x_0, y_0)$ lie on both $E_1$ and $E_2$; substituting into either of their equations yields

$$\left( \frac{a_1}{a_2} \right)^{\frac{m_1 m_2}{m_1-m_2}} + \left( \frac{b_1}{b_2} \right)^{\frac{m_1 m_2}{m_1-m_2}} = 1. \tag{3}$$

Thus (3) is a necessary and sufficient condition for $E_1$ and $E_2$ to be tangent at a point other than a vertex; the point of tangency in the first quadrant is given by (2). Moreover, $E_1$ lies inside [outside] $E_2$ if $m_1 > m_2$ [$m_1 < m_2$] (as is evident by considering Figures 1–3).

Before stating the main result, one more definition is needed. For $p \neq 0$, the *$p$–separation between* $(x_1, y_1)$ *and* $(x_2, y_2)$ is defined to be

$$\operatorname{sep}_p((x_1, y_1), (x_2, y_2)) := (|x_2 - x_1|^p + |y_2 - y_1|^p)^{\frac{1}{p}}. \tag{4}$$

We use this definition for descriptive ease only. For example, in Figure 1, we can say that the 2–separation between the endpoints of the segments is constant, since the 2–separation is the usual Euclidean distance. In Figure 2, we can say that the 1–separation between adjacent vertices of the ellipses is constant, as can be seen by examining (4) with $p = 1$.

We are now ready to state the main result.

**Theorem:** *Let $m_1, k > 0$ and $p \neq 0$ be given. Suppose that the $p$–separation between adjacent vertices of an $m_1$–ellipse is $k$. Then this $m_1$–ellipse is tangent to the $m_2$–ellipse*

$$|x|^{m_2} + |y|^{m_2} = k^{m_2},$$

*where $m_2$ is determined by*

$$\frac{1}{m_2} = \frac{1}{m_1} + \frac{1}{p}. \tag{5}$$

Before giving the straightforward proof, we look again at Figures 1–3. Figure 1 is an application of our Theorem with $m_1 = 1$ and $p = 2$. In this case, we are creating a family of diamonds (1–ellipses) with the property that the 2–separation between adjacent vertices is $k$; that is, the length of the sides of the diamonds is $k$. Since $m_2 = 2/3$ in this case, all such diamonds are tangent to the astroid

$$|x|^{2/3} + |y|^{2/3} = k^{2/3},$$

so that the envelope of a family of such diamonds is an astroid.

We also see that Figure 2 is an application of our Theorem with $m_1 = 2$ and $p = 1$. The symmetry of $m_1$ and $p$ in (5) implies that an astroid is *also* an envelope of a family of ellipses. Figure 3 illustrates the case $m_1 = p = 1$, with the expected $m_2 = \frac{1}{2}$. Of course there are infinitely many figures possible; further examples are given after the proof.

For the proof, let $m_1$, $k$, and $p$ be as described. Suppose that an $m_1$–ellipse $E(a, b; m_1)$ with $a, b > 0$ is such that the $p$-separation between adjacent vertices is $k$; that is,

$$a^p + b^p = k^p,$$

or equivalently,

$$\left(\frac{a}{k}\right)^p + \left(\frac{b}{k}\right)^p = 1. \tag{6}$$

Then by (3), this $m_1$–ellipse is tangent to $E(k, k; m_2)$ precisely when

$$\left(\frac{a}{k}\right)^{\frac{m_1 m_2}{m_1 - m_2}} + \left(\frac{b}{k}\right)^{\frac{m_1 m_2}{m_1 - m_2}} = 1. \tag{7}$$

Now considering the family of generic ellipses

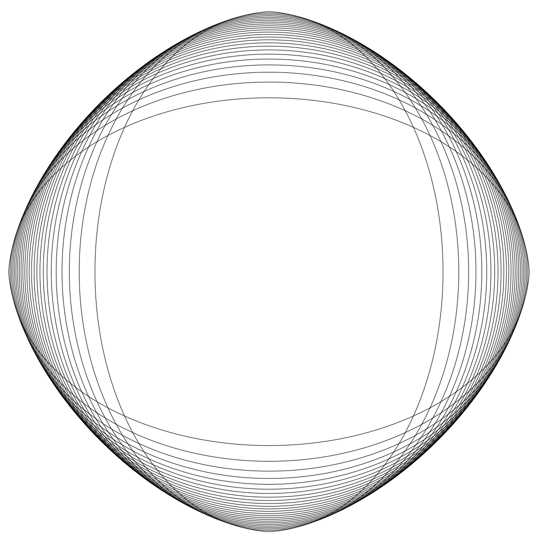
$$|x|^m + |y|^m = 1$$

for $m > 0$, it is evident that the point $(a/k, b/k)$ lies on precisely one such generic ellipse unless the point is a vertex, which is impossible when $a, b > 0$. Hence comparison of (6) and (7) results in

$$p = \frac{m_1 m_2}{m_1 - m_2},$$

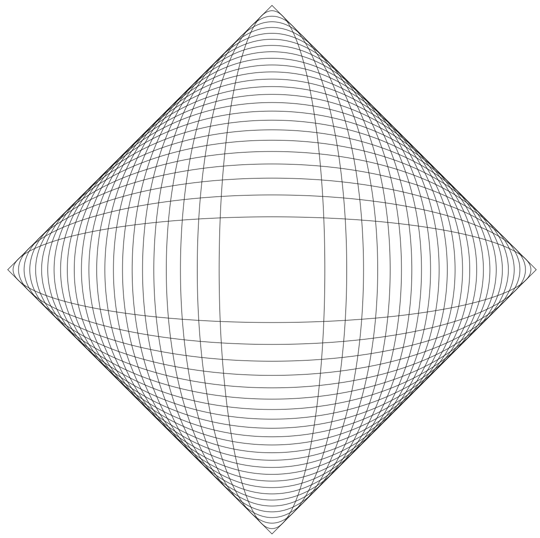
which is equivalent to (5). This completes the proof of the Theorem.

In Figures 4 and 5, we have $p \geq 1$; indeed, the knowledgeable reader will recognize (4) as the definition of a distance formula satisfying all the usual properties. However, (4) merely reflects a property of the vertices of a family of generic ellipses. So when $0 < p < 1$, as in Figure 6, we may still employ (4), being careful to interpret it as an algebraic relationship rather than a geometric one.



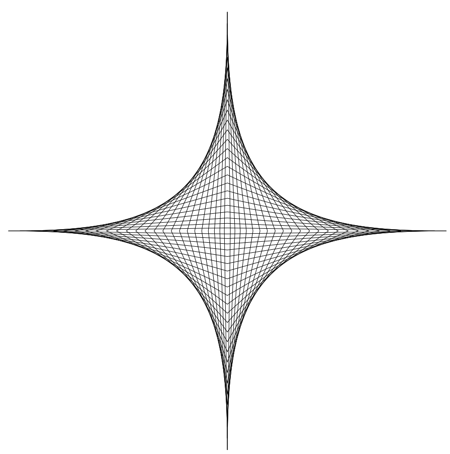
$m_1 = 2, \quad p = 8, \quad m_2 = \frac{8}{5}$

Figure 4



$m_1 = 2, \quad p = 2, \quad m_2 = 1$

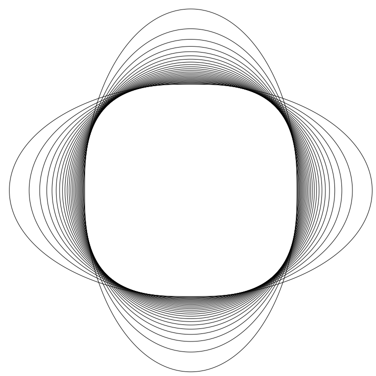
Figure 5



$m_1 = 1, \quad p = \frac{3}{4}, \quad m_2 = \frac{3}{7}$

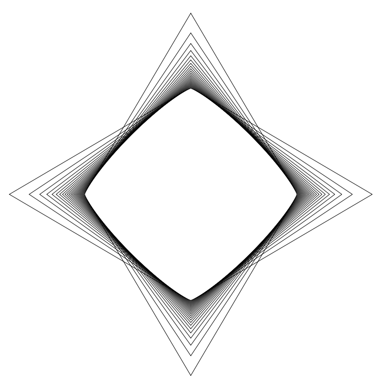
Figure 6

Note that since $p > 0$, we see from (5) that $m_1 > m_2$. Hence, each $m_1$-ellipse lies inside the $m_2$-ellipse. Interpreting (4) as a strictly algebraic relationship between vertices of a generic ellipse, why not allow $p < 0$? Several interesting envelopes result, as shown below. However, as we still require $m_1, m_2 > 0$, it follows from (5) that $m_1 < m_2$ in this case. Thus each $m_1$-ellipse lies *outside* the $m_2$-ellipse. (The careful reader will want to verify that the proof of the Theorem remains valid when $p < 0$; while the family of curves $|x|^m + |y|^m = 1$ for $m < 0$ no longer consists of generic ellipses, it is still the case that no two distinct such curves intersect.)



$$m_1 = 2, \quad p = -6, \quad m_2 = 3$$

Figure 7



$$m_1 = 1, \quad p = -6, \quad m_2 = \frac{6}{5}$$

Figure 8

It was the beauty of envelopes which first attracted them to the author's attention. Perhaps gone are the days when envelopes were created with pen and straightedge. But the advent of computer graphics allows for the creation of some truly striking images. It is hoped that the few included here might inspire others to explore this sublime geometrical world.

**References**

1. V. G. Boltyanskii, *Envelopes,* Macmillan, New York, 1964.

2. E. H. Lockwood, *A Book of Curves,* Cambridge University Press, New York, 1963.