Second-Order Recurrences with Nonconsecutive Initial Conditions

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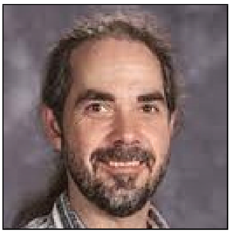

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Consider the following second-order recurrence relations:

$$G_{n+2} = G_{n+1} - G_n, \quad n \geq 0, \quad G_0 = G_6 = 1, \tag{1}$$

$$G_{n+2} = G_{n+1} - G_n, \quad n \geq 0, \quad G_0 = 1,\ G_6 = 2, \tag{2}$$

$$G_{n+2} = 2G_{n+1} - G_n, \quad n \geq 0, \quad G_0 = G_6 = 1. \tag{3}$$

These are not usual exercises in a discrete mathematics course—the initial conditions are nonconsecutive. Solving second-order recurrences with nonconsecutive initial conditions introduces a few surprises. For example, to solve (1), choose your favorite number for G_1 (such as 42), and write the resulting sequence:

$$1, 42, 41, -1, -42, -41, 1, 42, \ldots.$$

This will force $G_6 = 1$, and so (2) has *no* solution. But (3) *does* have a unique, if pedestrian, solution: the constant sequence 1.

The main purpose of this paper is to decide when a recurrence with nonconsecutive initial conditions has no solution, a unique solution, or infinitely many solutions. Throughout, we assume recurrences are linear, second-order, and homogeneous with constant, real coefficients and real initial conditions.

Preliminaries

Recall that the solution to the second-order linear recurrence relation given by

$$G_{n+2} = rG_{n+1} + sG_n, \quad n \geq 0, \tag{4}$$

with λ_1 and λ_2 being roots of the characteristic equation $\lambda^2 - r\lambda - s = 0$, is given by

$$G_n = c_1\lambda_1^n + c_2\lambda_2^n \tag{5}$$

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when $\lambda_1 \neq \lambda_2$, and

$$G_n = (c_1 + c_2 n)\lambda_1^n \tag{6}$$

when $\lambda_1 = \lambda_2$. The constants c_1 and c_2 are determined by the initial conditions, that is, values (usually) for G_0 and G_1. Note that λ_1, λ_2, c_1, and c_2 may be complex. (Of course, the reals may be considered as a subset of the complex numbers, but we use the term "complex" specifically to mean non-real.) It is easy to see from (4) that consecutive initial conditions (not necessarily G_0 and G_1) uniquely determine G. (We consider a sequence to be a function whose domain is the natural numbers, so that "G" refers to this function, while "G_n" refers to a specific term in the sequence. As is usual, we write "G_n" instead of "$G(n)$.")

We now consider the most general form of (4), where any two initial conditions may be specified:

$$G_{n+2} = rG_{n+1} + sG_n, \quad n \geq 0, \quad G_p, G_q \text{ given}, \quad 0 \leq p < q. \tag{7}$$

To help motivate the main result, we consider two examples that possess features characteristic of recurrences with nonconsecutive initial conditions.

Example 1. Consider the recurrence

$$G_{n+2} = 4G_n, \quad n \geq 0, \quad G_0 = 1, \quad G_4 = 16.$$

Note that the roots of the characteristic equation $\lambda^2 - 4 = 0$ are both real and are negatives of each other. Also note that if $G_0 = 1$, then G_4 *must* be 16. Moreover, G_1 is arbitrary; the sequence G has the form

$$1, \quad G_1, \quad 4, \quad 4G_1, \quad 16, \quad 16G_1, \ldots.$$

Example 2. Consider the recurrence

$$G_{n+2} = 2G_{n+1} - 4G_n, \quad n \geq 0, \quad G_0 = 1, \quad G_6 = 64.$$

The characteristic equation $\lambda^2 - 2\lambda + 4 = 0$ has complex conjugate roots $1 \pm \sqrt{3}i$. The sequence G begins

$$1, \quad G_1, \quad 2G_1 - 4, \quad -8, \quad -8G_1, \quad -16G_1 + 32, \quad 64, \ldots,$$

so that G_3 and G_6 must be -8 and 64, respectively. With $G_1 = 0$, we have the sequence

$$1, \quad 0, \quad -4, \quad -8, \quad 0, \quad 32, \quad 64, \ldots,$$

where powers of 2 seem to be lurking. In fact, the sequence $2^{-n}G_n$ is the periodic sequence

$$1, \quad 0, \quad -1, \quad -1, \quad 0, \quad 1, \quad 1, \quad 0, \ldots.$$

Quasi-periodicity

We now make an assumption that will hold for the remainder of this paper: $s \neq 0$. If $s = 0$, the recurrence is actually first-order and results in a geometric sequence. Of course, second-order recurrences may produce geometric sequences, but our discussion will not be adversely affected as long as $s \neq 0$ in these cases.

Rather than state the result first, we provide motivation by attempting to solve (7). We first look at solutions of the form (6), where $\lambda_1 = \lambda_2$. In this case, the initial conditions produce the system

$$G_p = (c_1 + c_2 p)\lambda_1^p,$$
$$G_q = (c_1 + c_2 q)\lambda_1^q.$$

The general solution to this system is given by

$$c_1 = \frac{qG_p\lambda_1^q - pG_q\lambda_1^p}{(q-p)\lambda_1^{p+q}}, \quad c_2 = \frac{G_p\lambda_1^q - G_q\lambda_1^p}{(p-q)\lambda_1^{p+q}}, \tag{8}$$

so we may uniquely solve for c_1 and c_2 for any two given initial values. Thus, we see that no subtleties arise when the characteristic equation has a repeated root.

Now assume that $\lambda_1 \neq \lambda_2$, so that (5) applies. The initial conditions require solving the system

$$G_p = c_1\lambda_1^p + c_2\lambda_2^p, \tag{9}$$
$$G_q = c_1\lambda_1^q + c_2\lambda_2^q. \tag{10}$$

The general solution to this system is given by

$$c_1 = \frac{G_q\lambda_2^p - G_p\lambda_2^q}{\lambda_1^q\lambda_2^p - \lambda_1^p\lambda_2^q}, \quad c_2 = \frac{G_p\lambda_1^q - G_q\lambda_1^p}{\lambda_1^q\lambda_2^p - \lambda_1^p\lambda_2^q}. \tag{11}$$

This presents no difficulty as long as

$$\lambda_1^q\lambda_2^p - \lambda_1^p\lambda_2^q \neq 0.$$

When $\lambda_1^q\lambda_2^p - \lambda_1^p\lambda_2^q = 0$ and $\lambda_2 \neq 0$ (which follows from the assumption $s \neq 0$), this is equivalent to

$$\left(\frac{\lambda_1}{\lambda_2}\right)^{q-p} = 1.$$

Note that this implies

$$\begin{aligned} G_q &= c_1\lambda_1^q + c_2\lambda_2^q \\ &= c_1\lambda_1^p\lambda_1^{q-p} + c_2\lambda_2^p\lambda_2^{q-p} \\ &= c_1\lambda_1^p\lambda_1^{q-p} + c_2\lambda_2^p\lambda_1^{q-p} \\ &= \lambda_1^{q-p}(c_1\lambda_1^p + c_2\lambda_2^p) \\ &= \lambda_1^{q-p}G_p. \end{aligned} \tag{12}$$

Of course, p and q may be far apart, so we introduce the following definition, reminiscent of the usual definition for periodicity.

Definition. When $\lambda_1 \neq \lambda_2$, the sequence G is said to be **quasi-periodic** if there is an integer $Q > 0$ such that

$$\left(\frac{\lambda_1}{\lambda_2}\right)^Q = 1 \quad \text{and} \quad \left(\frac{\lambda_1}{\lambda_2}\right)^M \neq 1$$

for $0 < M < Q$. This integer Q is called the **quasi-period of** G.

Recall the assumption that $\lambda_1 \neq \lambda_2$. While it is possible to define quasi-periodicity so that the case $\lambda_1 = \lambda_2$ is included, this would unnecessarily complicate statements of other results. Since there is always a unique solution to (7) when $\lambda_1 = \lambda_2$, distinguishing quasi-periodicity in this case offers no additional insight into the nature of the solutions. Moreover, the only type of sequence in the case $\lambda_1 = \lambda_2$ which could be called quasi-periodic would be geometric (including constant) sequences, and these are also generated when $\lambda_1 \neq \lambda_2$.

Now suppose that G is a quasi-periodic sequence with quasi-period Q. Since λ_1 and λ_2 are solutions to a quadratic equation with real coefficients, then either λ_1 and λ_2 are both real, or else we must have $\lambda_2 = \overline{\lambda_1}$ (where $\bar{\lambda}$ is the complex conjugate of λ).

We examine the real case first. Since G is quasi-periodic, we have $\lambda_1 \neq \lambda_2$, and hence it must be the case that $\lambda_2 = -\lambda_1$. Thus $\lambda_1/\lambda_2 = -1$, and therefore $Q = 2$. Without loss of generality, assume that $\varrho = \lambda_1 > 0$, so we may write

$$G_n = \varrho^n(c_1 + (-1)^n c_2). \tag{13}$$

Example 1 illustrates this case. Note that because $\lambda_2 = -\lambda_1$, we must have $r = 0$ (where r is as in (4)). Thus, Example 1 is essentially the only type of quasi-periodic sequence whose characteristic equation has real roots.

Proceeding to the complex case, put $\varrho = |\lambda_1|$ and find θ so that

$$\lambda_1 = \varrho e^{i\theta}, \quad \lambda_2 = \varrho e^{-i\theta}.$$

In Example 2, we have $\varrho = 2$ and $\theta = \pi/3$. (Note: Here and in the remainder of the paper, we choose θ to be the smallest positive angle satisfying a given condition.) Then

$$\left(\frac{\lambda_1}{\lambda_2}\right)^Q = \left(\frac{e^{i\theta}}{e^{-i\theta}}\right)^Q = e^{2iQ\theta} = \cos(2Q\theta) + i\sin(2Q\theta) = 1.$$

Note that $Q = 3$ in Example 2.

Because $\cos(2Q\theta) = 1$ and $\sin(2Q\theta) = 0$, it must be that $2Q\theta$ is some positive integer multiple k of 2π, and hence

$$\theta = \frac{k\pi}{Q}. \tag{14}$$

Therefore we may write G as follows, remembering that $c_2 = \overline{c_1}$ and using the notation $\Re z$ for the real part of the complex number z:

$$\begin{aligned} G_n &= c_1\lambda_1^n + c_2\lambda_2^n = c_1\lambda_1^n + \bar{c}_1(\overline{\lambda_1})^n \\ &= 2\,\Re(c_1\lambda_1^n) = 2\,\Re\left(c_1\varrho^n e^{in\theta}\right) \\ &= 2\varrho^n \Re\left(c_1 e^{in\theta}\right). \end{aligned} \tag{15}$$

From this representation, we have the following result, justifying the term "quasi-periodic" for describing G.

Lemma. *Suppose that G is quasi-periodic with quasi-period Q. Then with* $\varrho = |\lambda_1|$ *and k as given in* (14)*, the sequence*

$$H_n = \varrho^{-n} G_n$$

is periodic with period Q if either λ_1 *and* λ_2 *are real or k is even, and period* $2Q$ *if k is odd.*

Proof. Note that in the real case, the result follows immediately from (13). For the complex case, the result follows directly from (15), along with the observation that

$$e^{i(n+Q)\theta} = e^{in\theta} e^{iQ\theta} = e^{in\theta} e^{ik\pi},$$

which follows from the definition of θ (14). The minimality of Q in the definition of quasi-periodicity implies that H cannot have a period smaller than Q (or $2Q$ if k is odd). ■

Main result

Now that we have articulated the feature (namely, quasi-periodicity) of recurrence relations with nonconsecutive initial conditions, which may be problematic in their solution, we are able to state the main result.

Theorem. *Consider the recurrence relation*

$$G_{n+2} = rG_{n+1} + sG_n, \quad n \geq 0, \quad G_p, G_q \text{ given}, \quad 0 \leq p < q,$$

where $s \neq 0$*, and* λ_1, λ_2 *are the roots of the characteristic equation* $\lambda^2 - r\lambda - s = 0$*.*
Then we have:

1. *There is no solution if G is quasi-periodic with quasi-period Q,* $q - p$ *is a multiple of Q, and*

$$G_q \neq \lambda_1^{q-p} G_p;$$

2. *There is exactly one solution if G is not quasi-periodic, or if G is quasi-periodic with quasi-period Q, and* $q - p$ *is not a multiple of Q;*
3. *There are infinitely many solutions if G is quasi-periodic with quasi-period Q,* $q - p$ *is a multiple of Q, and* $G_q = \lambda_1^{q-p} G_p$*.*

Proof. First, suppose that G is not quasi-periodic. If $\lambda_1 = \lambda_2$, we see from (8) that c_1 and c_2 are uniquely determined, while if $\lambda_1 \neq \lambda_2$, we see from (11) that c_1 and c_2 are uniquely determined.

Now suppose that G is quasi-periodic with quasi-period Q. The case when λ_1 and λ_2 are real follows easily from considering Example 1 and (13). We are left with the case when the roots are complex.

Assume first that $q - p$ is a multiple of Q. Then

$$\left(\frac{\lambda_1}{\lambda_2}\right)^{q-p} = \left(\frac{\lambda_1}{\lambda_2}\right)^{Q} = 1,$$

so that, from (12), we must have

$$G_q = \lambda_1^{q-p} G_p.$$

Thus, if the initial conditions do not satisfy this equation, there is no solution to the recurrence. But if the initial conditions *do* satisfy this equation, the system (9, 10) is dependent. In this case, (10) is obtained by multiplying (9) by λ_1^{q-p}. Thus, specifying a value for, say, G_{p+1} will determine the sequence (see (1) for an example).

If $q - p$ is not a multiple of Q, then the denominators in (11) are not 0, so that c_1 and c_2 may be uniquely determined. ■

Quasi-periodic integer sequences

As an application of these ideas, we ask the following question: When is the solution to a second-order recurrence a quasi-periodic integer sequence? Several interesting ideas arise in considering an answer to this question.

It may seem intuitive that if G is a sequence of integers, then both r and s must be integers—but the question is more subtle. The solution to the recurrence

$$G_{n+2} = \frac{5}{2}G_{n+1} - G_n, \quad G_0 = 1, \quad G_1 = 2,$$

is given by $G_n = 2^n$. However, it may be shown that if the solution to a second-order recurrence is *not* a geometric sequence, then r and s must be integers when G is an integer sequence. The proof relies on Fatou's Lemma, and is included in the appendix for completeness.

So assume that G is a quasi-periodic integer sequence. It should be clear from the preceding remarks that if $r = 0$ and s is an integer greater than 0, then a quasi-periodic sequence is obtained as long as the initial data are suitable. This exhausts the case when λ_1 and λ_2 are real.

When λ_1 and λ_2 are complex, we note that

$$r = \lambda_1 + \lambda_2 = 2\varrho\cos\theta,$$

where ϱ and θ are defined as before. But we know that $\varrho^2 = \lambda_1\overline{\lambda_1} = -s$, so that

$$\cos^2\theta = \frac{r^2}{4\varrho^2} = -\frac{r^2}{4s}.$$

It is well known that if θ is a rational multiple of π (see (14)) and $\cos^2\theta$ is rational, then

$$\cos^2\theta \in \left\{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\right\}.$$

(A proof of this fact is given in [**1**].) This is sufficient to determine s in terms of r and $\cos^2\theta$, so that all possibilities may be found.

For completeness, we enumerate all quasi-periodic and periodic integer sequences generated by second-order recurrences. Periodic sequences are found by requiring $\varrho = 1$, so that $G = H$ (where H is as described in the theorem), and hence G is periodic. Note that the case $\cos^2\theta = 1$ is not addressed, as this implies that $\lambda_1 = \lambda_2$; this issue was discussed earlier. We also assume that initial conditions G_0 and G_1 are given, and that G_0 and G_1 are not both 0, as this generates the constant sequence 0 in all cases. There are just four types:

$$G_{n+2} = sG_n, \tag{16}$$

$$G_{n+2} = rG_{n+1} - r^2G_n, \tag{17}$$

$$G_{n+2} = rG_{n+1} - \frac{r^2}{2}G_n, \qquad r \text{ even}, \tag{18}$$

$$G_{n+2} = rG_{n+1} - \frac{r^2}{3}G_n, \qquad r \text{ divisible by 3}. \tag{19}$$

- Regarding (16), the recurrence $G_{n+2} = sG_n$ is quasi-periodic for any integer s. When $s > 0$, then λ_1 and λ_2 are real, and G has quasi-period 2. Note that G is geometric if $sG_0^2 = G_1^2$. When $s = 1$, then G is periodic with period 2 unless $G_0 = G_1$, in which case G is constant. When periodic, G has the form

$$G_0, G_1, G_0, \ldots.$$

When $s < 0$, then λ_1 and λ_2 are complex conjugates, which corresponds to the case $\cos^2\theta = 0$ and G has quasi-period 4. When $s = -1$, then G is periodic with period 4 and has the form

$$G_0, G_1, -G_0, -G_1, G_0, \ldots.$$

- The sequence given in (17) corresponds to the case $\cos^2\theta = 1/4$. Here, we have $s = -r^2$. When $r > 0$, we have $\cos\theta = 1/2$ and hence $\theta = \pi/3$. Then G has quasi-period 6. When $r = 1$, we see that G is periodic with period 6 and has the form

$$G_0, G_1, G_1 - G_0, -G_0, -G_1, G_0 - G_1, G_0, \ldots.$$

When $r < 0$, then $\cos\theta = -1/2$ and $\theta = 2\pi/3$. In this case, G has quasi-period 3. When $r = -1$, then G is periodic with period 3 and has the form

$$G_0, G_1, -G_0 - G_1, G_0, \ldots.$$

- The case $\cos^2\theta = 1/2$ gives (18), so that $\theta = \pi/4$ or $\theta = 3\pi/4$. In this case, $s = -r^2/2$, and G has quasi-period 8. Here, ϱ cannot be 1.
- Finally, (19) corresponds to the case $\cos^2\theta = 3/4$, and hence we have $\theta = \pi/6$ or $\theta = 5\pi/6$. In this case, $s = -r^2/3$ and G has quasi-period 12. Also here, ϱ cannot be 1.

Concluding remarks

We have seen that second-order recurrences with nonconsecutive initial conditions provide an interesting, yet accessible alternative to the usual problems offered in a discrete mathematics course. We hope that these results will inspire others. It particular, an interesting REU project might be to explore higher-order cases, both in deciding the nature of the solutions to higher-order recurrences, as well as finding all quasi-periodic and periodic integer sequences produced by such recurrences.

Appendix

Here, we wish to establish that if a second-order recurrence is given by (4) that is *not* a geometric sequence, and if G is a sequence of integers, then r and s must be integers.

This may be shown using Fatou's Lemma (see [**2**], 605, 629), stated below as it applies to our discussion.

Lemma (Fatou). *Suppose that*

$$g(x) = \sum_{n=0}^{\infty} G_n x^n,$$

with integer coefficients G_n, may be written as a rational function $\widetilde{P}(x)/\widetilde{Q}(x)$, where $\widetilde{P}(x)$, $\widetilde{Q}(x) \in \mathbb{Q}[x]$. Then we can write $g(x) = P(x)/Q(x)$, where P and Q are relatively prime polynomials with integer coefficients such that $Q(0) = 1$.

We apply Fatou's Lemma by first finding the generating function for G using the usual techniques. (For the purposes of the proof, it is not necessary to determine the interval of convergence.)

Proceeding, we have

$$\begin{aligned}
g(x) &= \sum_{n=0}^{\infty} G_n x^n \\
&= G_0 + G_1 x + \sum_{n=2}^{\infty} G_n x^n \\
&= G_0 + G_1 x + \sum_{n=2}^{\infty} (rG_{n-1} + sG_{n-2}) x^n \\
&= G_0 + G_1 x + rx \sum_{n=2}^{\infty} G_{n-1} x^{n-1} + sx^2 \sum_{n=2}^{\infty} G_{n-2} x^{n-2} \\
&= G_0 + G_1 x + rx \sum_{n=1}^{\infty} G_n x^n + sx^2 \sum_{n=0}^{\infty} G_n x^n \\
&= G_0 + G_1 x + rx(g(x) - G_0) + sx^2 g(x).
\end{aligned}$$

Solving for $g(x)$ gives

$$g(x) = \frac{G_0 + x(G_1 - rG_0)}{1 - rx - sx^2}.$$

Now put $\widetilde{P}(x) = G_0 + x(G_1 - rG_0)$ and $\widetilde{Q}(x) = 1 - rx - sx^2$. Hence, $\widetilde{P}(x)$ and $\widetilde{Q}(x)$ must be relatively prime, or else $\widetilde{P}(x)$ is a factor of $\widetilde{Q}(x)$, so that $g(x)$ is the generating function for a geometric series, contradicting our assumption that G is not geometric. Then

$$\frac{P(x)}{Q(x)} = \frac{\widetilde{P}(x)}{\widetilde{Q}(x)},$$

or equivalently, $P(x)\widetilde{Q}(x) = \widetilde{P}(x)Q(x)$. Since $P(x)$ and $Q(x)$ are relatively prime by Fatou's Lemma, we must have that $Q(x)$ divides $\widetilde{Q}(x)$, and since $\widetilde{P}(x)$ and $\widetilde{Q}(x)$ are relatively prime, we must have that $\widetilde{Q}(x)$ divides $Q(x)$. Thus, $Q(x)$ is a rational multiple of $\widetilde{Q}(x)$.

Since both $Q(0) = 1$ (by Fatou's Lemma) and $\widetilde{Q}(0) = 1$, we must therefore have that $Q(x) = \widetilde{Q}(x)$. Since we know from Fatou's Lemma that $Q(x)$ must have integer coefficients, we see that r and s must be integers.

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Summary. When consecutive initial conditions for second-order linear homogeneous recurrence relations with constant coefficients are given, the resulting sequence is uniquely determined. However, if the initial conditions are not consecutive, it may be the case that no sequence is possible, or that infinitely many sequences satisfy the recurrence.

References

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Proof Without Words: Ensphering Three Capped Prisms

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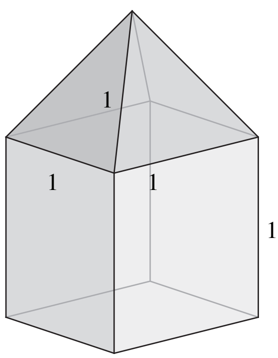

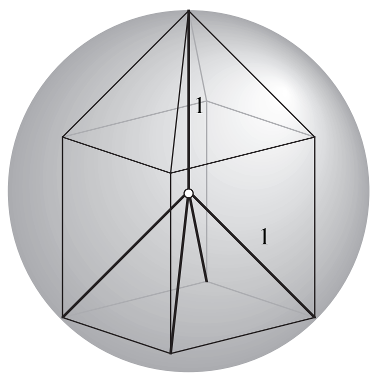

Summary. For $n = 3$, 4, or 5, a unit sphere is the smallest sphere that encloses a right prism with square sides on a regular n-gon capped by a regular pyramid with all sides of unit length. The points of incidence are the apex of the pyramid and the opposite n vertices of the prism.

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